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ON RADIATIVE SHOCK STRUCTURE

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By

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ABSTRACT

A study is made of plane gas dynamic and transverse magnetogasdynamic shock waves in fully ionized hydrogen, at conditions such that radiative processes are significant. An established steady state flow is found to exist for these waves only in astrophysical cases, since the wave thicknesses in such a flow are quite large. In the case of such an established flow the Rankine-Hugoniot jump equation (i.e., the relationship between the flow variables in the initial and final states) is analyzed and then solved numerically. It is shown that of the twelve roots of this equation only two are physical, corresponding to the supersonic initial and subsonic final states. The differential equations which describe the above shock waves are analyzed. The result is a set of four simultaneous nonlinear ordinary first order differential equations, the solution of which gives the structure of the shock, i.e., the dependence of the flow variables on position within the shock wave. The structure equations are solved numerically for cases where the shock is optically thick, i.e., the mean free path for absorption of radiation is much smaller than the characteristic lengths for change of the flow variables. This is called the Eddington or diffusive approximation. A new criterion is given for the validity of this approximation in terms of the shock itself, independent of the properties of the atmosphere. Numerical results of radiative shock structure in this approximation are given.

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NOMENCLATURE

Symbol	Meaning	First Occurrence
a	Stefan-Boltzmann Constant (= 7.67×10^{-15} cgs units)	Chap. II
A	Ion atomic weight	Chap. II
A	Shock strength constant $(= \frac{2Qm}{P^2} - 1)$	Chap. II
A_k	An arbitrary column vector	App. C
$B_v(T)$	The Planck Black Body Function	Chap. II
$B_v'(\theta)$	Dimensionless Black Body Function	Chap. II
B_k^m	Constant in the moment approximation expansion	Chap. IV
c	Speed of light (= 3×10^{10} cm/sec)	Chap. II
C_v	Specific heat at constant volume	Chap. II
C_o	A constant	Chap. II
C_k^m	Constant in the moment approximation expansion	Chap. IV
D	Function of $\gamma(= \frac{\gamma - 1}{2})$	Chap. II
dm	Element of mass	Chap. II
ds	Infinitesimal path length	Chap. II

Symbol	Meaning	First Occurrence
$d\Sigma$	Element of area	Chap. II
$d\Omega$	Element of solid angle	Chap. II
e	Electron charge ($= 4.803 \times 10^{-10}$ esu)	Chap. II
$\vec{E}$	Electric field vector	Chap. II
E	Magnitude of $\vec{E}$	Chap. II
e_ν	Coefficient of true emission for frequency ν	Chap. II
f	Function of $\gamma (= \frac{2(3\gamma - 4)}{\gamma - 1})$	Chap. III
$\vec{F}$	Radiative flux vector	Chap. II
F	Magnitude of $\vec{F}$	Chap. II
F'	Dimensionless radiative flux	Chap. II
$F(\omega)$	Function of ω	Chap. III
$F_1(\omega)$, $F_\gamma(\omega)$	Functions of ω	App. A
$F(\omega, \theta)$	Function of ω and θ	Chap. V
F_θ	$\frac{\partial}{\partial \theta} F(\omega, \theta)$	Chap. V
F_ω	$\frac{\partial}{\partial \omega} F(\omega, \theta)$	Chap. V
$\lvert F \rvert_{max}$	Maximum absolute value of $F(\omega, \theta)$ on the shock curve	Chap. V
$F(y)$	Line vector, function of the column vector y	Chap. V
$F_i(y)$	Component of $F(y)$	App. C

Symbol	Meaning	First Occurrence
g	Function of $\gamma (= \frac{\gamma + 1}{\gamma - 1})$	Chap. III
g'	Gaunt factor	Chap. II
$G(\omega)$, $G_o(\omega)$	Functions of ω	Chap. III
$G_1(\omega)$	Function of ω	App. A
$G(\omega, \theta)$	Function of ω and θ	Chap. V
G_θ	$\frac{\partial}{\partial \theta} G(\omega, \theta)$	Chap. V
G_ω	$\frac{\partial}{\partial \omega} G(\omega, \theta)$	Chap. V
$\lvert G \rvert_{max}$	Maximum absolute value of $G(\omega, \theta)$ on the shock curve	Chap. V
G_{max}	Maximum value of $G(\omega, \theta)$ on the curve $F(\omega, \theta) = 0$	App. D
h	Planck's constant $(= 6.625 \times 10^{-27}$ erg-sec)	Chap. II
h	Function of $\gamma (= \frac{2 - \gamma}{\gamma - 1})$	Chap. III
h	Increment in ω	App. D
$\vec{H}$	Magnetic field vector	Chap. II
H	Magnitude of $\vec{H}$	Chap. II
h'	Perturbation in magnetic field	App. B
i, j, k, m, n, p, q	Integers	
I	Radiative intensity	Chap. II
J	Radiative source function	Chap. II

Symbol	Meaning	First Occurrence
j_ν	Coefficient of emission for frequency ν	Chap. II
k	Boltzmann's constant (= 1.3804×10^{-16} erg/^{o}K)	Chap. II
k	Increment in θ	App. D
K	Root locus parameter	Chap. III
k_c	Thermal conductivity	Chap. II
ℓ, m, n	Direction cosines	Chap. II
L_H	Energy transfer characteristic length	Chap. II
L_M	Electromagnetic characteristic length	Chap. II
L_P	Momentum transfer characteristic length	Chap. II
L_Q	Diffusion approximation characteristic length	Chap. V
L_R	(Rosseland) radiative mean free path	Chap. II
$\bar{L}_P$	Mean of L_P through the shock	Chap. V
$\bar{L}_Q$	Mean of L_Q through the shock	Chap. V
$\bar{L}_R$	Mean of L_R through the shock	Chap. V
$L(y)$	Diagonal characteristic length matrix	Chap. V

Symbol	Meaning	First Occurrence
$L_h(\omega)$	Function of ω	App. A
m	Mass flow constant	Chap. II
M	Mach number	Chap. III
M_m	Minimum obtainable Mach number	Chap. III
M	A matrix	Chap. V
M_{ij}	Component of matrix M	Chap. V
m_s	Mass per scatterer	Chap. II
m_H	Mass of H^+ ion	Chap. V
m_e	Mass of an electron	Chap. V
$M_g(\omega)$	Function of ω	App. A
n	Number of particles per cc	Chap. II
n_a	Number of absorbers per cc	Chap. II
n_e	Number of electrons per cc	Chap. II
n_i	Number of ions per cc	Chap. II
N	An integer denoting order of the moment approximation	Chap. IV
N_o	Avogadro's number ($= 6.025 \times 10^{23}$ mole^{-1})	Chap. II
N_o	Number of zeros of an algebraic expression	App. A
N_p	Number of poles of an algebraic expression	App. A

Symbol	Meaning	First Occurrence
N_P	Prandtl Number	Chap. V
$\mathcal{O}$	Order of	Chap. V
p	Thermodynamic pressure	Chap. II
p_r	Radiation pressure	Chap. II
p_r'	Dimensionless radiation pressure	Chap. II
p_t	Total (thermodynamic and radiation) pressure	App. B
p'	Perturbation in total pressure	App. B
p_e	Electron pressure	Chap. II
P	Momentum flow constant	Chap. II
$p(\lambda)$, $q(\lambda)$	Polynomials in λ	App. C
$p(\cos\theta)$	Radiative phase factor	Chap. II
$p^o(\ell, \ell')$	Radiative phase factor	Chap. II
P_o, P_1	Points on the phase plane	Chap. V
$P_n(\ell)$	The n'th Legendre Polynomial in ℓ	Chap. IV
Q	Energy flow constant	Chap. II
$\vec{r}$	Radius vector	Chap. II
R	Gas constant	Chap. II
R, R_o, R_1	Regions on the real ω axis	App. A
R_o	Rydberg constant ($= 109{,}737.31\ \text{cm}^{-1}$)	Chap. II
r_o, r_1, s_o, s_1	Numbers of eigenvalues of given sign	App. C

Symbol	Meaning	First Occurrence
S	Radiation strength parameter ($= aP^7/m^8R^4$)	Chap. II
S	Entropy	App. B
t	Time	Chap. II
T	Absolute temperature	Chap. II
t	Perturbation in temperature	Chap. III
t_y	Shock thickness for the variable y	Chap. V
t_θ	Shock thickness for temperature	Chap. V
t_ω	Shock thickness for velocity	Chap. V
U	Thermal energy density per unit mass	Chap. II
U_r	Radiation energy density per unit volume	Chap. II
U_r'	Dimensionless radiation energy	Chap. II
$\vec{v}$	Velocity vector	Chap. II
v, V	Magnitude of $\vec{v}$	Chap. II
v	Perturbation in velocity	Chap. III
v'	Perturbation in velocity	App. B
w_o	Albedo	Chap. II
x,y,z	Cartesian coordinates	Chap. II
x*	A value of x	Chap. V
X	Density ratio across the shock	Chap. III

Symbol	Meaning	First Occurrence
y	Column vector	Chap. V
y^i	Component of column vector y	App. C
y	Denotes one of the shock variables	Chap. V
Z	Magnetic strength parameter $\left(= \frac{E^2 m^2}{\mu P^3}\right)$	Chap. II
α	Sound speed	App. B
α	An infinitesimal	Chap. III
β	Velocity parameter $\left(= \frac{P}{mc}\right)$	Chap. II
β	Alfven speed	App. B
γ	Specific heat ratio	Chap. II
γ'	Equivalent specific heat ratio for signal propagation	Chap. III
δ	An infinitesimal	Chap. III
δ	Mean characteristic length ratio	Chap. V
δ'	Characteristic length ratio	Chap. V
δ_{nm}	Kronecker delta	Chap. IV
Δy	Maximum absolute value of y along the shock curve	Chap. V
ϵ	Radiation parameter	Chap. II
ϵ_h	Magnetic field parameter	Chap. III
η	Viscosity	Chap. II

Symbol	Meaning	First Occurrence
θ	Dimensionless temperature	Chap. II
θ	Angle between incident and scattered radiation	Chap. II
κ	Mass absorption coefficient	Chap. II
λ_D	Debye length	Chap. II
$\lambda, \lambda_1, \lambda_2$	Eigenvalues	Chap. V
λ_k	Eigenvalue corresponding to A_k	App. C
λ^s	Eigenvalue used to determine initial slope of shock curve	App. C
μ	Magnetic permeability	Chap. II
ν	Frequency	Chap. II
ρ	Density	Chap. II
ρ'	Perturbation in density	App. B
σ	Electrical conductivity	Chap. II
σ	Radiative cross section	Chap. II
τ	Optical depth	Chap. II
ϕ	Azimuthal angle	Chap. II
ϕ	Dimensionless radiative intensity	Chap. II
ϕ_n	Coefficient of $P_n(\ell)$ in expansion for ϕ	Chap. IV
$\phi_{\nu 1}^{(m)}$	Coefficient of $L_{R\nu}^m$ in expansion for $\phi_{\nu 1}$	Chap. V

Symbol	Meaning	First Occurrence
χ	Dimensionless magnetic field	Chap. II
ψ	Function of n and T	Chap. V
ψ'	Function of T	Chap. V
ω	Dimensionless velocity	Chap. II
ω_c	Location of "center of gravity" of singularities	App. A
ω_f^-, ω_f^o, ω_f^1	Roots of $F(\omega)$	Chap. II
$\overline{\omega}_f^o$, $\overline{\omega}_f^1$	Roots of $F_1(\omega)$	Chap. II
ω_g^-, ω_g^o, ω_g^1	Roots of $G(\omega)$	Chap. II
$\overline{\omega}_g^o$, $\overline{\omega}_g^1$	Roots of $G_1(\omega)$	Chap. II
ω_-', ω_o', ω_1'	Roots of $G(\omega) - F(\omega)$	App. A
ω_{oi}	Location of a root of an algebraic expression	App. A
ω_{pi}	Location of a pole of an algebraic expression	App. A
ω_γ^-, ω_γ^o, ω_γ^1	Roots of $F_\gamma(\omega)$	App. A

Sub- and Superscripts (unless otherwise noted)

0 Pre-shock state

1 Post-shock state

a Absorptive part

s Scattering part

ν In the frequency interval $(\nu, \nu + d\nu)$

CHAPTER I

INTRODUCTION AND HISTORICAL SURVEY

In this paper gas dynamic and transverse magnetohydrodynamic shocks in fully ionized hydrogen are studied, under the following assumptions:

(1) The shock is plane and infinite;

(2) Continuum theory (the MHD equations) can be used;

(3) Radiative energy content and radiative transfer have significant effect on the phenomena;

(4) A steady flow is established in the sense that a "shock frame" (a frame of reference, moving with the shock velocity, in which all time derivatives are zero) exists;

(5) The gas can be considered ideal;

(6) Radiation intensity is attenuated in the gas by absorption rather than by scattering;

(7) The atmosphere is in "local thermodynamic equilibrium," i.e., Kirchoff's law of radiation applies.

The differential equations resulting from the above seven assumptions are considered at their equilibrium points, and the resulting "jump equation" is analyzed and solved numerically to find the states upstream and downstream of the shock.

It is shown that the jump equation, although generally of twelfth degree, has only two physical solutions, corresponding to the pre-shock and post-shock states. Numerical results are presented showing the temperature, pressure and density ratios across the shock as functions of two dimensionless shock parameters.

The solution of the differential equations themselves requires two additional assumptions, namely:

(8) The angular dependence of the radition field is such that the radiative intensity can be adequately represented by the first two terms of a Fourier series in the Legendre polynomials.

(9) The radiative mean free path is smaller than the characteristic distances in which all the fluid properties vary.

With these additional assumptions a numerical integration of the shock differential equations can be performed in the gas dynamic case. However, assumptions 8 and 9, along with assumption 4, restrict the validity of the solution to a certain limited range of physical conditions. The characteristic length for change in temperature in the specific cases considered varies from about 10 miles to about .05 light years, so that the assumption of established flow, which depends on the shock thickness being considerably smaller than the characteristic length of the physical environment, cannot hold in any laboratory generated shock, and probably not under any planetary conditions. The results may, however, have astrophysical as well as theoretical interest.

Assumption 9, which leads to the "Eddington Approximation," further restricts the limits of applicability of the present solution to a certain range of values of initial shock conditions. A criterion is proposed for the validity of assumption 9, and for certain cases where it does hold the numerical solutions to the differential equations are presented.

Of the other assumptions, only number 6, that of an absorptive atmosphere, is at all restrictive, in that it sets an upper limit of about $10^{7\circ}$ K to the temperatures that can be considered. There is much controversy as to the validity of assumption 2, the use of the continuum model, in gas dynamic shock structure theory. However, there is some experimental evidence (Ref. 1) which tends to support the use of this model. Furthermore, continuum theory is at least self-consistent in the present problem, since the shocks are at least several mean free paths in thickness.

The first derivations of the jump conditions across a gas dynamic shock wave were made by Rankine (Ref. 2) and Hugoniot (Ref. 3), the latter derivation also showing that the shock equations had only two equilibrium points, corresponding to the upstream and downstream states. The problem of the solution of the gas dynamic shock structure equations, which are in their general form a pair of simultaneous first order nonlinear differential equations, was also first undertaken by Rankine in his original paper. In this work, the problem was reduced to a pair of equations, one algebraic and the other differential, by considering the gas to be inviscid with finite thermal conductivity. Rayleigh (Ref. 4) showed that this reduction of the order of the equations was supportable only for sufficiently weak shocks, and performed a similar reduction considering zero thermal conductivity and finite viscosity, finding no such restriction in that case. The problem of gas dynamic shock wave structure with finite viscosity and finite thermal conductivity was first undertaken by Becker (Ref. 5), and was considered in its most general form by Gilbarg and Paolucci (Refs. 6 and 7). Reference 7 also contains a strong argument for the use of continuum theory in shock structure problems. Other authors, notably Wang Chang (Ref. 8), Mott Smith (Ref. 9), Zoller (Ref.10) and Grad (Ref. 11) have undertaken the solution of the problem by direct application of the Boltzmann equation, holding that the continuum results are a poor first approximation of the real solution.

When the fluid is electrically conducting and subjected to a magnetic field, in other words in the magnetogasdynamic case, the problem of shock wave structure becomes much more complicated. In its general form, four simultaneous first order nonlinear differential equations must be integrated between equilibrium points. When the normal component of the magnetic field is zero, i.e., for transverse shocks, the number of equations is reduced to three. The problem of magnetogasdynamic shock structure has been studied notably by Marshall (Ref. 12), Burgers (Ref. 13) Ludford (Ref. 14), Germain (Ref. 15) and Anderson (Ref. 16). In none of these works, however, is a numerical integration of more than two simultaneous equations attempted, the order of the problem being reduced by the assumption that certain of the transport coefficients are zero. The difficult problem of the integration of more than two simultaneous first order differential equations between their equilibrium points is discussed theoretically in Refs. 16 and 17. The jump equation for transverse MHD shocks, although more complicated than the gasdynamic Rankine-Hugoniot equation, is not overly difficult to solve (Refs. 18 and 19).

It is shown in Ref. 20 that in considering the equation of state of an ionized gas there are many common cases in which radiation effects cannot be ignored. The shock structure and jump equations for these cases must thus be modified to include radiative energy storage and radiative transfer. The jump equations for radiative shocks were first discussed in the special case of very strong shocks by Sachs (Ref. 21), and in the case of non-established flow (when assumption 4 does not hold) by Guess and Sen (Ref. 22). A study of the jump equations for established flow with transverse magnetic field was made by Pai and Speth (Ref. 23). In the present paper, this jump equation is analyzed further, and certain general theorems are proven. It is shown that there exists only two physical solutions to the equation, and the approximate location of these roots is determined theoretically, so that Newton's method can be used directly in the numerical solutions of the jump equation. The weak shock and strong shock approximations are also examined, the latter leading to Sach's results. A discussion is contained which clarifies the arguments in Refs. 21 and 23 as to the validity of the established flow assumption.

The subject of radiative shock structure has been extensively studied since 1952, when Prokof'ev (Ref. 24) considered the case of established flow with zero viscosity and zero thermal conductivity, i.e., "radiation smoothing" alone. Prokof'ev's original work was corrected and expanded by Zeldovitch (Ref. 25), Raizer (Ref. 26), Clarke (Ref. 27), Heaslet and Baldwin (Ref. 28), and in the MHD case by Mitchner and Vinokur (Ref. 29). The influence of radiation on shock structure has been studied as a shock tube phenomenon, notably in the non-established case by Pomerantz (Ref. 30) and Olfe (Ref. 31). There is much recent work which, as in the present paper, considers the problem of radiative shock structure with viscous, heat conduction and radiation smoothing all present. Chow (Ref. 32) considers shocks in a transparent atmosphere, while Traugott (Ref. 33)and Scala and Sampson (Ref. 34) give results in both the thin (transparent) and thick (highly absorptive) cases. In the present paper, a complete solution is given for shocks in optically thick atmospheres. A set of equations is presented which is valid for all absorptive atmospheres. An important criterion is presented in terms of dimensionless shock parameters as to when an atmosphere can be considered thick with respect to a given shock, i.e., the validity of the diffusive approximation is shown to be a function of the shock itself rather than of the various mean free paths in the atmosphere.

CHAPTER II

PHYSICAL AND MATHEMATICAL FORMULATION OF THE PROBLEM

The importance of the radiative energy content of a gas as compared to its thermal energy content can be roughly gauged by the size of the parameter ϵ, defined as $aT^3/nk (= 55.6T^3/n)$, where a is the Stefan-Boltzmann constant (7.67×10^{-15} egs units), T is the temperature in ^{o}K, n is the number of particles per cc, and k is Boltzmann's constant (1.3804×10^{-16} erg/^{o}K). This parameter is the ratio of the radiative energy per unit volume, aT^4, to nkT, which is 2/3 of the thermal energy per unit volume in an ideal monatomic gas. For a gas at equilibrium, a value of $\epsilon = 3/2$ implies the equal importance of the two modes of energy storage, and a value of $\epsilon = 3$ implies the equality of the thermodynamic and radiation pressures. Thus, in writing the momentum equation for gases at states for which ϵ is near or greater than 1, it is necessary to include a term representing the gradient of radiation pressure, and the energy equation must contain terms representing the flux of radiant energy and the converted flow of radiant energy per unit volume.

Radiative processes can also be important in the direct transfer of energy from one volume of fluid to another volume. In this case, one must consider the relative size of the conductive energy flux $k_c \nabla T$, where k_c is the thermal conductivity, and the radiative energy flux $\vec{F}$. Since heat transfer requires spatial variations in the properties of the gas, nothing can be said a priori about the relative importance of conductive and radiative transfer. The appropriate differential equations must first be solved.

The problem to be considered here is that of a plane infinite shock wave propagating into a fully ionized gas at a given "upstream" state denoted by the subscript 0. The state behind the shock will be denoted by subscript 1. The shock has a velocity $\vec{V}$ in the x direction and, because it is plane, the fluid properties vary only with x. Radiative energy storage and transfer are assumed to be important (see Fig. 2.1).

In some aspects of this work (e.g., the "jump equations") it adds very little to the complexity of the problem to consider the presence of a magnetic field $\vec{H}$ in the y-z plane. Since the gas is ionized and therefore conductive, an electric field $\vec{E}$, also in the y-z plane and perpendicular to $\vec{H}$, may be present also. The MHD approximation can be used in this case.

A standard technique in shock structure problems is the writing of all the equations in the "shock frame," i.e., a frame of reference moving with the shock (at velocity $\vec{V}$) in

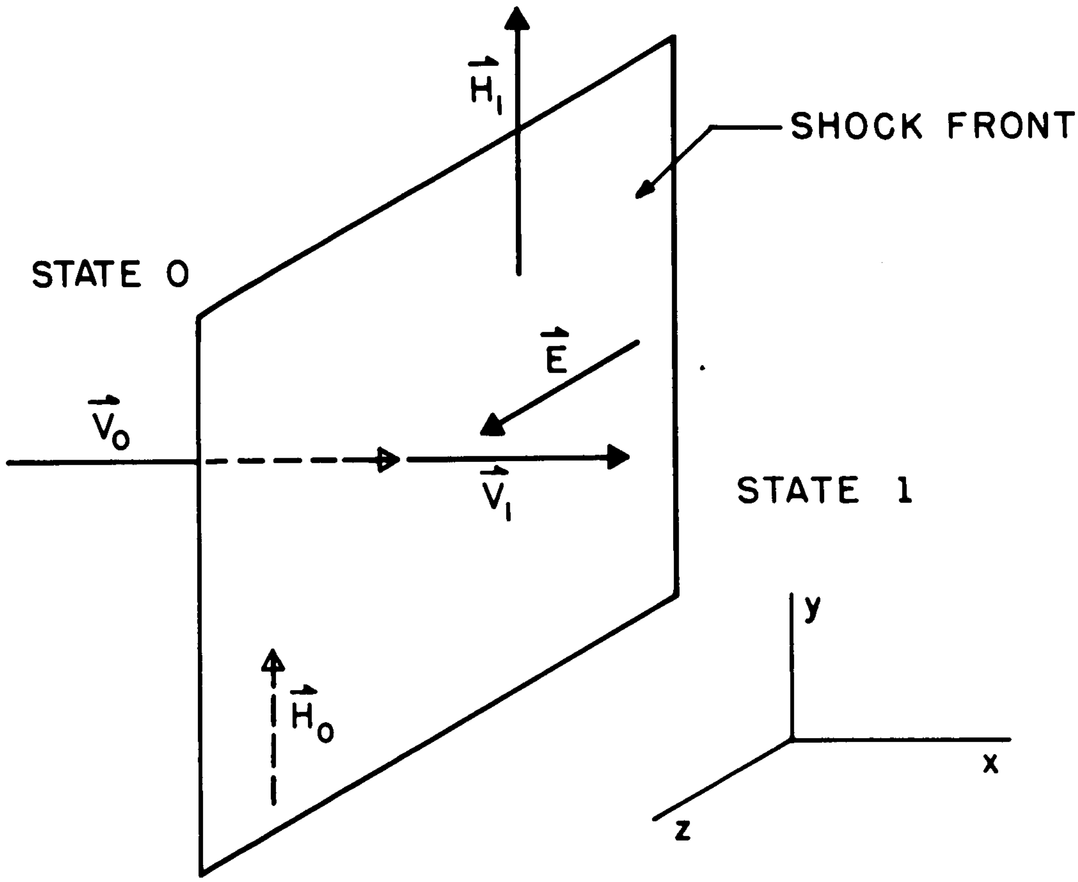

FIG. 2.1

which all time derivatives vanish. In fluid dynamic shocks, for which there is only one characteristic velocity, it is well established that such a frame exists. In this case, however, electromagnetic radiation, which has characteristic speed c, is also considered, and there is a question as to whether such a steady flow can ever be "established" in the sense that there is a frame in which all time derivatives vanish. In a shock tube, for instance, a visible glow from the upstream fluid will definitely reach a downstream observer before the shock front. Also, a portion of the energy from such a glow will be lost to the gas. If the characteristic length of the physical problem is considered to be infinite, no such loss would occur.

It will be assumed that a sufficient condition for the existence of a shock frame is that the radiation emitted by an element of the gas is reabsorbed by another element of the gas whose distance from the first element is not large compared to the physical characteristic length. In such a case, the radiative energy being transferred from hotter regions of the gas to colder regions can be considered as being at the same time convected along with the gas, in much the same way as the heat transferred from molecule to molecule by conduction is carried along by the molecules in their flow.

Assuming this to be the case, the fluid dynamic equations pertinent to the one-dimensional problem can be written, in the shock frame, as follows: (Refs. 7, 23)

Conservation of mass:

$$\frac{d}{dx}(\rho v) = 0 \tag{2.1}$$

Conservation of x-momentum:

$$\rho v \frac{dv}{dx} + \frac{d}{dx}(p + pr + \frac{1}{2} \mu H^2 - \frac{4}{3} \eta \frac{dv}{dx}) = 0 \tag{2.2}$$

Conservation of energy:

$$\rho v \frac{d}{dx}(U + \frac{U_r}{\rho} + \frac{1}{2} v^2 + \frac{p+p_r}{\rho}) + \frac{d}{dx}(EH - \frac{4}{3} \eta v \frac{dv}{dx} - k_c \frac{dT}{dx} + F) = 0 \tag{2.3}$$

where

ρ is the density of the gas

v is the velocity of the gas relative to the shock

p is the thermodynamic pressure

p_r is the radiation pressure, defined below

μ is the permeability of the gas (mks)

H is the magnitude of the magnetic field (mks)

E is the magnitude of the electric field (mks)

η is the viscosity of the gas

U is the thermal energy density per unit mass

U_r is the radiation energy density per unit volume, defined below

k_c is the thermal conductivity of the gas

T is the absolute temperature

F is the flux of radiative energy, defined below.

Integration of equation 2.1 gives:

$$\rho v = m \tag{2.4}$$

where m is a constant, called the mas-flow constant.

Integration and rearrangement of equation 2.2 gives:

$$mv + p + p_r + \frac{1}{2}\mu H^2 - \frac{4}{3}\eta\frac{dv}{dx} = P \tag{2.5}$$

where P is a constant, called the momentum-flow constant.

Integration and rearrangement of equation 2.3 gives:

$$Pv + mU + vU_r - \frac{mv^2}{2} + F + EH - \frac{\mu v H^2}{2} - k_c\frac{dT}{dx} = Q \tag{2.6}$$

where Q is a constant, called the energy-flow constant.

An additional differential equation is provided by the "combined Maxwell" equation of MHD:

$$\frac{1}{\mu\sigma}\frac{dH}{dx} = vH - \frac{E}{\mu} \tag{2.7}$$

where σ is the electrical conductivity of the gas. The electric field, being tangential, does not vary with x. Here the magnetic field is in the z direction and the electric field is in the y direction.

The equation of state of an ideal gas:

$$p = \rho RT \;;\; U = C_v T \tag{2.8}$$

will be used since, according to Kelly (Ref. 35) equation 2.8 is valid when the number of particles within a Debye sphere is much greater than 1. For a singly ionized gas with one species this criterion is:

$$\frac{4\pi}{3} n\lambda_D{}^3 \gg 1$$

where, the Debye length:

$$\lambda_D = \left(\frac{kT}{4\pi ne^2}\right)^{\frac{1}{2}}$$

where e is the charge of an electron. This inequality reduces to

$$\epsilon = \frac{aT^3}{nk} \gg 2.7 \times 10^{-7} \tag{2.9}$$

which will always be true if radiation is significant. According to Ref. 20, the gas will not be degenerate if

$$\epsilon T^{-3/2} \gg 1.5 \times 10^{-13} \tag{2.10}$$

Thus if $\epsilon = 0.01$, a small value of the radiation parameter, 2.8 will hold if $T \ll 10^8$.

The quantities p_r, U_r and F which appear in equations 2.5 and 2.6 make necessary the consideration of radiative transfer phenomena in order to solve the problem presented.

The specific radiative intensity $I_\nu(r; \ell, m, n; t)$ is defined (Ref. 36) such that $I_\nu \ell d\nu d\Sigma d\Omega dt$ is the amount of radiant energy in the frequency interval $(\nu, \nu + d\nu)$ which is transported across an element of area $d\Sigma$, in directions confined to an element of solid angle $d\Omega$, during a time dt. Here ℓ is the cosine of the angle which the direction considered makes with the outward normal to $d\Sigma$, and m and n are the other two direction cosines. $d\Sigma$ is located at the point $\vec{r}$ in space (see Fig. 2.2). In one-dimensional problems such as the present one (called "plane stratified" problems in radiative transfer terminology) where the properties of the atmosphere vary in the x direction only, I_ν can be considered a function of x, ℓ, ϕ, t and ν only, where ϕ is the azimuthal angle. Since in this problem all directions in the shock plane are equivalent (i.e., the radiation is axially symmetric) and it is assumed that a frame exists in which all time derivatives vanish, I_ν is a function of x, ℓ and ν only.

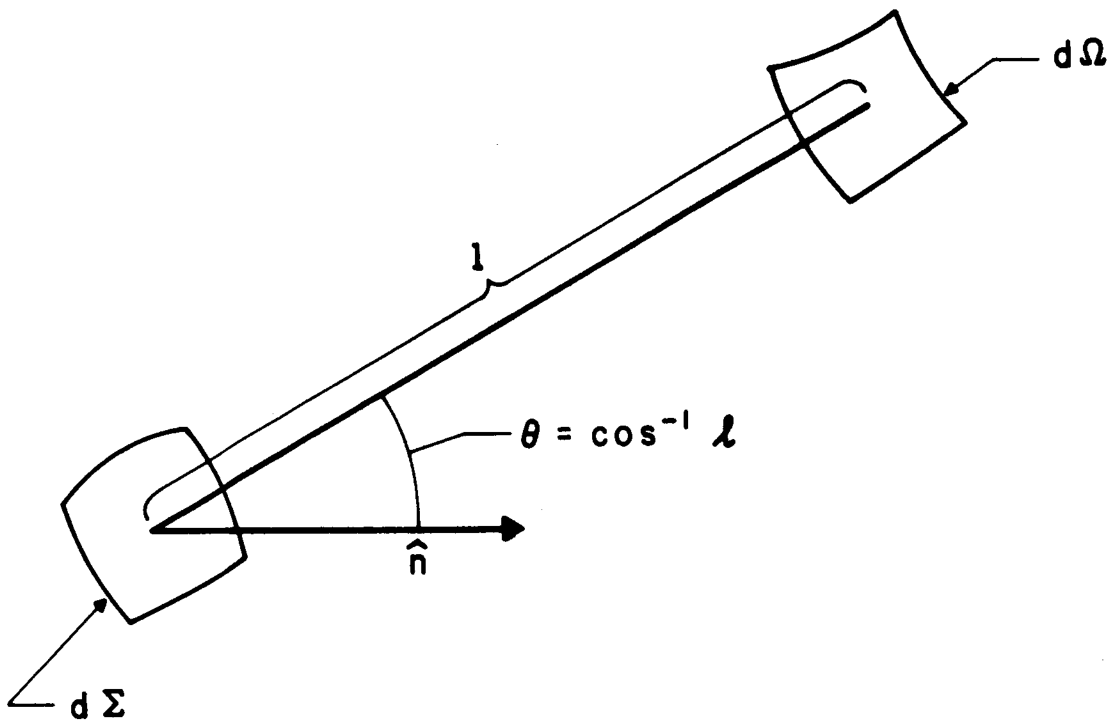

FIG. 2.2

The radiative mass absorption coefficient of a substance, κ_ν, is defined, in the plane parallel case, such that, if during its passage through a path length ds within the substance, a pencil of radiation of intensity I_ν is weakened by an amount δI_ν, then:

$$\delta I_\nu = \kappa_\nu \rho I_\nu ds$$

The "optical depth" τ_ν is defined such that:

$$d\tau_\nu = \kappa_\nu \rho dx$$

Thus:

$$\ell \delta I_\nu = I_\nu d\tau_\nu$$

κ_ν can be considered as consisting of two parts: $\kappa_\nu{}^a$, the coefficient of true absorption, and $\kappa_\nu{}^s$, the coefficient of scattering. The scattering cross section $\sigma_\nu{}^s$ is related to the mass-scattering coefficient $\kappa_\nu{}^s$ by

$$\sigma_\nu{}^s = m_s \kappa_\nu{}^s$$

where m_s is the density of the substance divided by the number of scattering centers per unit volume, i.e., the mass per scattering center. In the case of a singly ionized gas, for instance, the major scattering process for radiation of thermal wavelength is Thompson scattering from the electrons (Ref. 37), but since κ_ν^s is a mass-scattering coefficient, m_s in this case is the mass of the ions.

The differential mass scattering coefficient $d\kappa_\nu^s$ can be defined as:

$$d\kappa_\nu^s = \kappa_\nu^s p(\cos\,\theta)\frac{d\Omega'}{4\pi}$$

where θ is the angle between the incident and scattered radiation and p is the "phase function" normalized such that

$$\int p(\cos\,\theta)\frac{d\Omega'}{4\pi} = 1$$

the integration being performed over all directions of scattered radiation.

The albedo, w_o, of a gas is defined such that

$$\kappa_\nu^s = \kappa_\nu w_o$$

$$\kappa_\nu^a = \kappa_\nu(1 - w_o)$$

w_o is thus a measure of the fraction of the absorbed radiation which will reappear as scattered radiation.

The radiative emission coefficient j_ν is defined (Ref. 36) such that an element of mass dm emits or scatters into directions confined to an element of solid angle $d\Omega$, in the frequency interval $(\nu, \nu + d\nu)$ and in time dt, an amount of radiant energy given by

$$j_\nu dm d\Omega d\nu dt$$

The radiative source function J_ν is defined as $J_\nu = j_\nu/\kappa_\nu$. J_ν can be considered as being made up of two parts, emissive and scattering:

$$J_\nu(\tau, \ell) = e_\nu/\kappa_\nu + \frac{1}{2} w_o \int_{-1}^{1} I_\nu(\ell')p^o(\ell, \ell')d\ell'$$

where: it is assumed that the atmosphere is plane stratified; e_ν is the coefficient of true emission, defined similarly to j_ν, but not including scattering; and $p^o(\ell, \ell')$ is the phase factor, now defined in terms of the direction cosines ℓ and ℓ' of the scattered and incident radiation, respectively. The integral represents the radiation in the frequency interval, incident from all directions, which is scattered into the spherical sector $(\ell, \ell + d\ell)$.

The equation of radiative transfer can be written (Ref. 36) in terms of the above-defined quantities as:

$$\ell \frac{\partial I_\nu(\tau, \ell)}{\partial \tau_\nu} = J_\nu - I_\nu \qquad (2.11)$$

The amount of radiant energy in the frequency interval $(\nu, \nu + d\nu)$ which is transferred across $d\Sigma$ during time dt is given by (Ref. 36)

$$d\nu \, d\Sigma \, dt \int I_\nu \ell \, d\Omega$$

where the integration is over all solid angles. The net radiative flux, i.e., the normal flow of radiant energy per unit area per unit frequency interval per unit time, is thus:

$$F_\nu = \int I_\nu \ell \, d\Omega$$

or, in axially symmetric cases:

$$F_\nu = 2\pi \int_{-1}^{1} \ell I_\nu \, d\ell \tag{2.12}$$

F_ν is the x component of the vector $\vec{F}_\nu = [F_\nu, 0, 0]$.

The radiative energy density, $U_{r\nu} d\nu$, in the frequency interval $(\nu, \nu + d\nu)$ at any given point is (Ref. 36) the amount of radiant energy per unit volume in that interval which is in course of transit in the immediate neighborhood of that point. In the axi-symmetric case:

$$U_{r\nu} = \frac{2\pi}{c} \int_{-1}^{1} I_\nu \, d\ell \tag{2.13}$$

Finally, a radiative stress tensor can be defined, of which, in this case, only the diagonal component $p_{r\nu}$, the "radiation pressure," need be considered:

$$p_{r\nu} = \frac{2\pi}{c} \int_{-1}^{1} \ell^2 I_\nu d\ell \tag{2.14}$$

Integrated radiative quantities are defined as the frequency-dependent quantities integrated over ν from 0 to ∞:

$$I(\tau, \ell) = \int_0^\infty I_\nu(\tau, \ell) d\nu \tag{2.15}$$

$$J = \int_0^\infty J_\nu d\nu = \int_0^\infty \frac{e_\nu}{\kappa_\nu} d\nu + \frac{1}{2} w_o \int_0^\infty \int_{-1}^{1} I_\nu(\ell') p^o(\ell, \ell') d\ell' d\nu$$

$$= \int_0^\infty \frac{e_\nu}{\kappa_\nu} d\nu + \frac{1}{2} w_o \int_{-1}^{1} I(\ell') p^o(\ell, \ell') d\ell' \tag{2.16}$$

$$\kappa = \int_0^\infty \kappa_\nu d\nu \tag{2.17}$$

$$d\tau = \rho dx \int_0^\infty \kappa_\nu d\nu = \rho\kappa dx \tag{2.18}$$

$$F = \int_0^\infty F_\nu d\nu \qquad = 2\pi \int_{-1}^{1} \ell I d\ell \tag{2.19}$$

$$U_r = \int_0^\infty U_{r\nu} d\nu \qquad = \frac{2\pi}{c} \int_{-1}^{1} I d\ell \tag{2.20}$$

$$p_r = \int_0^\infty p_{r\nu} d\nu \qquad = \frac{2\pi}{c} \int_{-1}^{1} \ell^2 I d\ell \tag{2.21}$$

Equation 2.11 can be integrated over frequency to give:

$$\frac{\ell}{\rho} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial I_\nu}{\partial x} d\nu = J - I \tag{2.22}$$

If κ_ν is independent of frequency, or can be approximated by some mean, then the equation of transfer is:

$$\ell \frac{\partial I}{\partial \tau} = J - I \tag{2.23}$$

where $d\tau = \kappa\rho dx = \frac{dx}{L_R}$ where L_R is the "radiative mean free path."

The relations thus far obtained, repeated here:

$$\rho v = m \tag{2.24}$$

$$mv + p + p_r + \frac{1}{2}\mu H^2 - \frac{4}{3}\eta \frac{dv}{dx} = P \tag{2.25}$$

$$Pv + mU + vU_r - \frac{mv^2}{2} + F + EH - \frac{\mu v H^2}{2} - k_c \frac{dT}{dx} = Q \tag{2.26}$$

$$\frac{1}{\mu\sigma} \frac{dH}{dx} = vH - \frac{E}{\mu} \tag{2.27}$$

$$p = \rho RT \; ; \; U = C_v T \tag{2.28}$$

$$F = 2\pi \int_{-1}^{1} \ell d\ell \int_0^\infty d\nu I_\nu \tag{2.29}$$

$$U_r = \frac{2\pi}{c} \int_{-1}^{1} d\ell \int_{0}^{\infty} d\nu I_\nu \tag{2.30}$$

$$p_r = \frac{2\pi}{c} \int_{-1}^{1} \ell^2 d\ell \int_{0}^{\infty} d\nu I_\nu \tag{2.31}$$

$$\ell \frac{\partial I_\nu}{\partial \tau_\nu} = J_\nu - I_\nu \tag{2.32}$$

are a set of 10 equations in the 11 unknowns ρ, v, p, H, p_r, U, U_r, F, T, I_ν and J_ν with parameters m, P, Q and E, and gas properties η, k_c, R, C_v, μ, σ and κ_ν. One more relation is needed for the solution of this set. This relation is available in equation 2.16, the definition of J_ν as soon as the gas to be considered and the range of physical conditions are specified.

The gas to be studied in this work will be completely ionized hydrogen, some of whose properties must now be examined. The specific heat ratio, γ, is 5/3, and the specific heat $C_v = R/\gamma - 1$. The gas constant R for a singly ionized gas is $2N_o k/A$, where N_o is Avogadro's number and A is the atomic weight of the ions. In the case of ionized hydrogen, $R = 2N_o k$. Discussion of the transport properties η, k_c and σ will be deferred until Chapter 5. The radiative properties of hydrogen, which have been studied in some detail (Ref. 38), must, however, be considered at this point in order that the appropriate approximation to equation 2.16 be found, and the set of equations be completed.

It will be assumed that the emissivity of the gas e_ν is the same as that of a gas at "local thermodynamic equilibrium," i.e., that, following Kirchhoff's law:

$$\frac{e_\nu}{\kappa_\nu} = B_\nu(T)$$

where

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

is the Planck black body function, h being Planck's constant. According to Ref. 39 this assumption is valid whenever the electron-electron collision frequency is much greater than the frequency of recapture collisions between electrons and ions, and this inequality is true for ionized hydrogen, where the recapture process is $H^+ + e \rightarrow H + h\nu$. Thus local thermodynamic equilibrium will hold. At the equilibrium point of equation 2.11, $I_\nu = J_\nu$, so that the frequency dependence of I_ν can be assumed to be approximately the same as that of $B_\nu(T)$, having a peak at or near the Wien frequency. For a fully ionized gas, the major scattering process at thermal frequencies is Thompson scattering (Ref. 37), and the major absorptive process is continuum (free-free) absorption. It will be assumed that the physical conditions are such that the albedo $w_o \ll 1$. For this to hold true it is sufficient that the mean free path for Thompson scattering be much larger than that for bremstrahlung, i.e.,

$$L_R^{\,s} \gg L_R^{\,a}$$

The cross section for Thompson scattering is 6.65×10^{-25} cm^2 (Ref. 37) and the mean free path is thus

$$L_R^{\,s} = \frac{1}{n\sigma^s} = \frac{1.502 \times 10^{24}}{n} \qquad (2.33$$

where n is the number of electrons per cc.

Menzel and Perkeris (Ref. 38) give the absorption coefficient per ionized hydrogen atom as:

$$\sigma_\nu^{\,a} = \frac{C_o p_e}{\nu^3 T^{3/2}}(e^{h\nu/kT} - 1)g' \qquad (2.34)$$

where C_o is a constant equal to 2.67×10^{24} in cgs units, p_e is the electron pressure, and g' is a function of ν and T given by:

$$g' = 1 - 0.1728\left(\frac{\nu}{R_o}\right)^{1/3}\left(1 - \frac{2kT}{h\nu}\right)$$

R_o being the Rydberg constant.

The appropriate mean absorption coefficient for this problem is the Rosseland mean (see Chapter V) defined as:

$$\sigma^a = \frac{\int_0^\infty \frac{\partial}{\partial T} B_\nu(T)\,d\nu}{\int_0^\infty \frac{1}{\sigma_\nu^a} \frac{\partial}{\partial T} B_\nu(T)\,d\nu}$$

Using equation 2.34 for σ_ν^a:

$$\sigma^a = 3.32 \times 10^{-7} \frac{p_e}{T^{9/2}}\left[1 - .1098\left(\frac{T}{1.57 \times 10^5}\right)^{1/3}\right] cm^2 \quad (2.35)$$

(Ref. 38)

The electron pressure p_e is $n_e kT_e$, where n_e is the number of electrons per cc and T_e is the electron temperature. For fully ionized hydrogen n_e is equal to n_i the number of ions per cc: $n_e = n_i = n$. It will be assumed that the electron temperature is equal to the gas temperature T, so that:

$$\sigma^a = 3.32 \times 10^{-7} \frac{nk}{T^{7/2}}\left[1 - .1098\left(\frac{T}{1.57 \times 10^5}\right)^{1/3}\right] cm^2 \qquad (2.36)$$

The radiative mean free path for absorption is:

$$L_R^a = \frac{1}{\rho\kappa^a} = \frac{1}{n_a\sigma^a} = \frac{1}{n_a m_a \kappa^a}$$

where m_a is the mass per absorber and n_a is the number of absorbers, in this case n, the number of electrons per cc. Then:

$$L_R{}^a = \frac{1}{n\sigma^a} \tag{2.37}$$

$$L_R{}^a = \frac{1}{3.32 \times 10^{-7}k} \frac{T^{7/2}}{n^2} \frac{1}{1 - .1098\left(\frac{T}{1.57 \times 10^5}\right)^{1/3}} \text{ cm}$$

$$= 2.18 \times 10^{22} \frac{T^{7/2}}{n^2} \frac{1}{1 - .1098\left(\frac{T}{1.57 \times 10^5}\right)^{1/3}} \text{ cm} \tag{2.38}$$

Thus, the albedo is much less than one if:

$$2.18 \times 10^{22} \frac{T^{7/2}}{n^2} \frac{1}{1 - .1098\left(\frac{T}{1.57 \times 10^5}\right)^{1/3}} \ll \frac{1.502 \times 10^{24}}{n}$$

or

$$\frac{T^{7/2}}{n} \frac{1}{1 - .1098\left(\frac{T}{1.57 \times 10^5}\right)^{1/3}} \ll 68.8 \tag{2.39}$$

This can be expressed as:

$$\epsilon T^{1/2} \frac{1}{1 - .1098\left(\frac{T}{1.57 \times 10^5}\right)^{1/3}} \ll 3825$$

Thus, for instance, for $\epsilon = 1$, the inequality 2.39 will hold for $T < 10^{7\,o}K$, and for $T = 10^{6\,o}K$, it will hold for $\epsilon < 3$, i.e., for $n > 5 \times 10^{19}/cc$. At lower temperatures, 2.39 is true for all reasonable number densities.

If the inequality 2.39 holds, the source function becomes:

$$J_\nu = B_\nu(T) \tag{2.40}$$

or, integrated over frequency:

$$J = \int B_\nu(T) d\nu = \frac{caT^4}{4\pi}$$

Equation 2.40 is the equation required to complete the set.

It is convenient for the work that follows to condense and non-dimensionalize the equations. The set can be reduced to four equations by substituting equations 2.24, 2.28, 2.29, 2.30, 2.31 and 2.40 into the rest. However, two of these equations would be integro-differential, involving integrals over ℓ and ν. It is thus convenient to retain equations 2.29, 2.30 and 2.31 and to use as the reduced set the following seven equations:

$$m(v + \frac{RT}{v}) + p_r + \frac{1}{2}\mu H^2 - \frac{4}{3}\eta \frac{dv}{dx} = P \tag{2.42}$$

$$Pv + \frac{m}{\gamma-1} RT + vU_r - \frac{mv^2}{2} + F + EH - \frac{\mu v H^2}{2} - k_c \frac{dT}{dx} = Q \qquad (2.43)$$

$$vH - \frac{1}{\mu\sigma}\frac{dH}{dx} = \frac{E}{\mu} \qquad (2.44)$$

$$\ell \frac{\partial I_\nu}{\partial \tau_\nu} = B_\nu(T) - I_\nu \qquad (2.45)$$

$$F = 2\pi \int_0^\infty d\nu \int_{-1}^{1} \ell I_\nu d\ell \qquad (2.46)$$

$$U_r = \frac{2\pi}{c} \int_0^\infty d\nu \int_{-1}^{1} I_\nu d\ell \qquad (2.47)$$

$$P_r = \frac{2\pi}{c} \int_0^\infty d\nu \int_{-1}^{1} \ell^2 I_\nu d\ell \qquad (2.48)$$

These seven equations can be non-dimensionalized to give:

$$L_p \frac{d\omega}{dx} = \frac{\theta}{\omega} + \omega + p_r' + \frac{1}{2}\frac{\chi^2}{Z} - 1 \qquad (2.49)$$

$$L_H \frac{d\theta}{dx} = \theta + 2D\omega U_r' - D[(1-\omega)^2 + A] + D\chi(2 - \frac{\chi\omega}{Z}) + \frac{2D}{\beta} F' \qquad (2.50)$$

$$L_m \frac{d\chi}{dx} = \omega\chi - Z \qquad (2.51)$$

$$L_{R\nu}\, \ell \frac{\partial \phi_\nu}{\partial x} = B_\nu'(\theta) - \phi_\nu \qquad (2.52)$$

$$F' = \frac{1}{2}\int_0^\infty d\nu \int_{-1}^{1} \ell \phi_\nu d\ell = \frac{1}{2}\int_{-1}^{1} \ell\phi d\ell \qquad (2.53)$$

$$U_r' = \frac{1}{2}\int_0^\infty d\nu \int_{-1}^{1} \phi_\nu d\ell = \frac{1}{2}\int_{-1}^{1} \phi d\ell \tag{2.54}$$

$$P_r' = \frac{1}{2}\int_0^\infty d\nu \int_{-1}^{1} \ell^2\phi_\nu d\ell = \frac{1}{2}\int_{-1}^{1} \ell^2\phi d\ell \tag{2.55}$$

where:

$$\omega = \frac{mv}{P}$$

$$\theta = \frac{m^2RT}{P^2}$$

$$\chi = \frac{Em}{P^2} H$$

$$\phi_\nu = \frac{4\pi I_\nu}{cP}$$

$$A = \frac{2QM}{P^2} - 1$$

$$Z = \frac{E^2m^2}{\mu P^3}$$

$$D = \frac{\gamma - 1}{2}\left(= \frac{1}{3} \quad \text{for} \quad \gamma = \frac{5}{3}\right)$$

$$\beta = \frac{P}{mc}$$

$$B_\nu'(\theta) = \frac{4\pi}{cP} B_\nu\left(\frac{P^2\theta}{m^2R}\right)$$

$$\int_0^\infty d\nu B_\nu'(\theta) = S\theta^4$$

$$\int_{0}^{\infty} \phi_{\nu} d\nu = \phi$$

$$S = \frac{aP^{7}}{m^{8}R^{4}}$$

$$L_{p} = \frac{4}{3}\frac{\eta}{m}$$

$$L_{H} = \frac{k_{c}}{C_{v}m}$$

$$L_{m} = \frac{m}{P\mu\sigma}$$

Equations 2.49 - 2.55 will be studied in the following chapters.

CHAPTER III

JUMP EQUATIONS

An essential step in the solution of the shock structure equations, and a study of much interest in itself, is the determination of the "equilibrium points" of the equations, i.e., those values of the variables for which all derivatives are equal to zero. If the shock is considered as a transition between equilibrium states of the gas, then the equilibrium points will correspond to these states. In gas-dynamic shocks two equilibrium points, corresponding to the "pre-shock" or "upstream" state and the "post-shock" or "downstream" state exist, and these states are found by solving the Rankine-Hugoniot equations. It will be shown below that the equations of Chapter 2 also have two equilibrium points, and numerical solutions to the equations corresponding to the ordinary Rankine-Hugoniot solutions will also be given.

If all the derivatives in equations 2.49 - 2.55 are set equal to zero, it is seen, first of all, that:

$$\phi_\nu = B'_\nu(\theta) \tag{3.1}$$

and thus is independent of ℓ. Then:

$$F' = \frac{1}{2} S\theta^4 \int_{-1}^{1} \ell d\ell = 0 \tag{3.2}$$

$$U_r' = \frac{1}{2} S\theta^4 \int_{-1}^{1} d\ell = S\theta^4 \qquad (3.3)$$

$$p_r' = \frac{1}{2} S\theta^4 \int_{-1}^{1} \ell^2 d\ell = \frac{1}{3} S\theta^4 \qquad (3.4)$$

Also:

$$\frac{\theta}{\omega} + \omega + \frac{1}{3} S\theta^4 + \frac{1}{2}\frac{\chi^2}{Z} - 1 = 0 \qquad (3.5)$$

$$\theta + 2D\omega S\theta^4 - D[(1 - \omega)^2 + A] + D\chi(2 - \frac{\chi\omega}{Z}) = 0 \qquad (3.6)$$

$$\omega\chi - Z = 0 \qquad (3.7)$$

Equations 3.5 to 3.7 can be combined into two:

$$\theta(1 + \frac{1}{3}\epsilon) = \omega(1 - \omega) - \frac{1}{2}\frac{Z}{\omega} \qquad (3.8)$$

$$\theta(1 + 2D\epsilon) = D[(1 - \omega)^2 + A - \frac{Z}{\omega}] \qquad (3.9)$$

where $\epsilon \equiv S\omega\theta^3$ (3.10)

The parameter ϵ is, in terms of the dimensional variables, $\frac{aT^3}{\rho R}$, i.e., approximately the ratio of the equilibrium radiation energy to thermal energy of the gas.

Equations 3.8 and 3.9 (using the definition 3.10) can be rearranged into:

$$(f\theta)^4 = \frac{K}{\omega}\left[(1 - \omega)(1 - g\omega) + A + \frac{hZ}{\omega}\right] \equiv \frac{KG(\omega)}{\omega^2} \qquad (3.11)$$

$$f\theta = (1 - \omega)(7\omega - 1) - A - \frac{2Z}{\omega} \equiv -\frac{F(\omega)}{\omega} \qquad (3.12)$$

where:

$$f(\gamma) = 2\left(\frac{3\gamma - 4}{\gamma - 1}\right) \qquad (3.13)$$

$$g(\gamma) = \frac{\gamma + 1}{\gamma - 1} \qquad (3.14)$$

$$h(\gamma) = \frac{2 - \gamma}{\gamma - 1} \qquad (3.15)$$

$$K = \frac{3f^3}{S} \qquad (3.16)$$

The functions of the specific heat ratio γ in equations 3.13 to 3.15 have been left in functional form even though in any physical situation to which they would apply, $\gamma = 5/3$.

Note that for $\gamma = 4/3$, which is the specific heat ratio for a photon gas (Ref. 40), the functions $F(\omega)$ and $G(\omega)$ are identically equal, and a mathematically degenerate case exists (see Appendix A).

For $\gamma = 5/3$

$$f(\gamma) = 3 \qquad (3.17)$$

$$g(\gamma) = 4 \tag{3.18}$$

$$h(\gamma) = \frac{1}{2} \tag{3.19}$$

and

$$K = 81/s \tag{3.20}$$

For $\gamma = 4/3$

$$f(\gamma) = 0 \tag{3.21}$$

$$g(\gamma) = 7 \tag{3.22}$$

$$h(\gamma) = 2 \tag{3.23}$$

and

$$K = 0 \tag{3.24}$$

The polynomials $F(\omega)$ and $G(\omega)$, defined in equations 3.12 and 3.11, respectively, are:

$$F(\omega) = \omega(1 - \omega)(1 - 7\omega) + (A\omega + 2Z) \tag{3.25}$$

$$G(\omega) = \omega(1 - \omega)(1 - g\omega) + (A\omega + hZ) \tag{3.26}$$

For $\gamma = 4/3$, $F(\omega) = G(\omega)$.

Equations 3.11 and 3.12 can be combined into a single polynomial equation in ω, which can be called the "jump equation" for radiative shocks:

$$\frac{\omega^2 G(\omega)}{(F(\omega))^4} = \frac{1}{K} \tag{3.27}$$

In gas-dynamic shocks, the jump, or Rankine-Hugoniot equation, one form of which is:

$$G_o(\omega) = (1 - \omega)(1 - g\omega) + A = 0 \qquad (3.28)$$

where $A = \frac{2Qm}{P^2} - 1$ as before, has solutions which depend on only one parameter A (aside from the fluid property g). The solutions of equation 3.27, on the other hand, depend on the three parameters A, S and Z. The jump equation for MHD shocks with the magnetic field in the plane of the shock can be written with two parameters A and Z, and equation 3.27 with $H = 0$ has two parameters A and S. Consideration of radiation thus adds an extra parameter to the jump equation. There is a simple reason for this: in ordinary gas dynamic shocks, the ratios of the state variables across the shock are functions of one dimensionless variable, say the initial Mach number, alone, and independent of the level of temperature or pressure. However, when radiation is considered, the temperature level, or, more exactly, the value of ϵ, at the initial and final states plays an important role in determining those states. Therefore, a second parameter, corresponding to ϵ in a similar way to that in which A corresponds to the Mach number, appears in the jump equation.

It can be shown (see Appendix A) that equation 3.27, although of 12th degree, has only two physically possible roots ω^o and $\omega^1 (\omega^o \geq \omega^1)$ which correspond to the pre- and post-shock states respectively. Figure 3.1 shows graphically the location of these roots. Curve A is a plot of

$$y = \omega(1 - \omega)(1 - 7\omega)$$

and curve B is a plot of

$$y = \omega(1 - \omega)(1 - g\omega)$$

for $g = 4(\gamma = 5/3)$. The straight line C is drawn with intercept $y = -2Z$ and slope -arc tan A. The straight line D has y intercept $-hZ$ and the same slope. Line E has y intercept 0 and the same slope. Referring to the figure, the pre-shock state has:

$$\omega_g^o \leq \omega^o \leq \omega_f^o$$

while the post-shock state has

$$\omega_f^1 \leq \omega^1 \leq \omega_g^1$$

When no magnetic field is present:

$$\overline{\omega}_g^o \leq \omega^o \leq \overline{\omega}_f^o$$

and

$$\overline{\omega}_f^1 \leq \omega^1 \leq \overline{\omega}_g^1$$

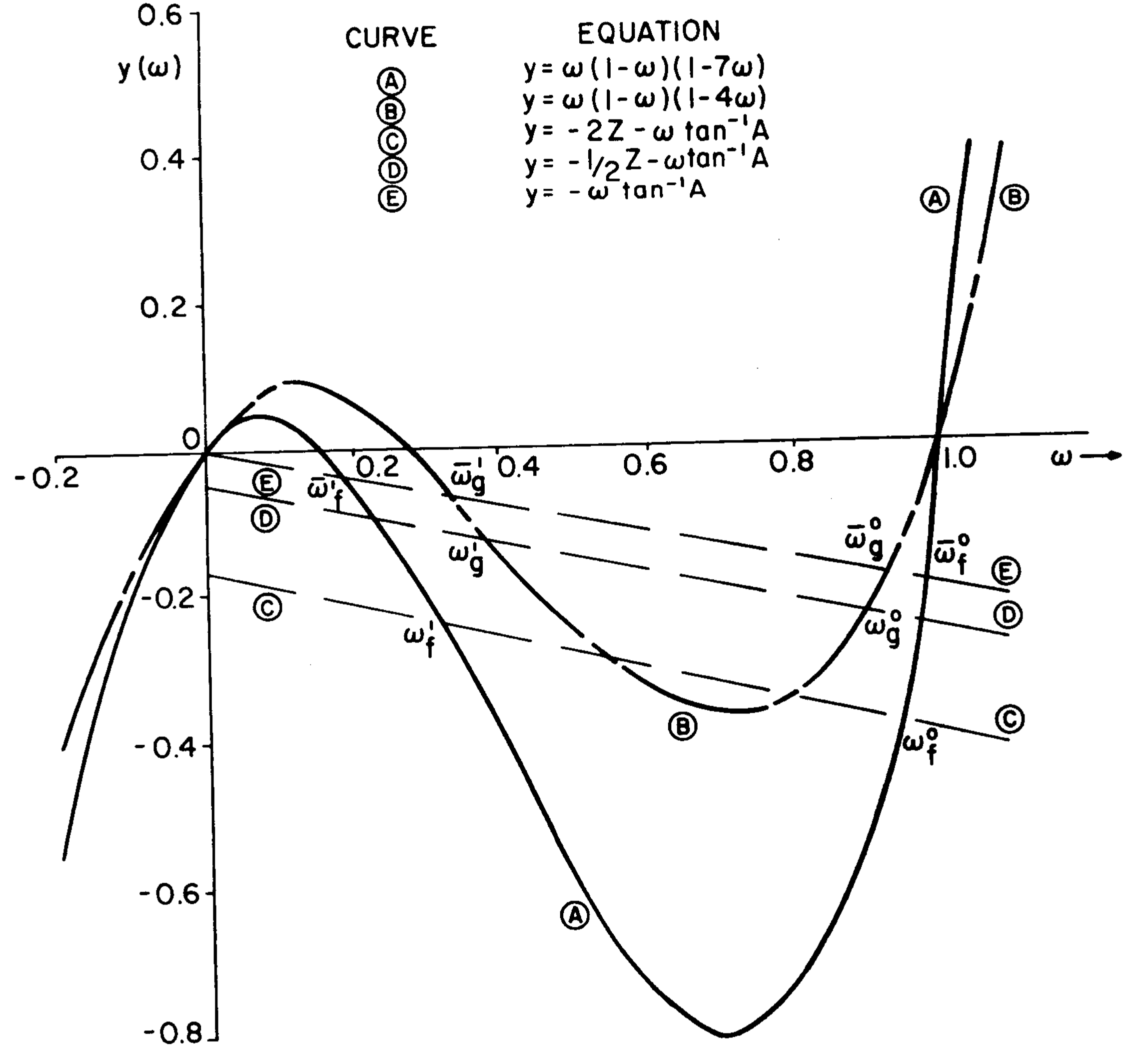

FIG. 3.1 LOCATION OF THE ROOTS OF THE JUMP EQUATION

$\overline{\omega}_g^o$ and $\overline{\omega}_g^1$ are the solutions to the gas-dynamic jump equation, 3.28. Summarizing the results of Appendix A as expressed in Fig. 3.1, the pre-shock ω, ω^o lies between the larger of the two positive roots of $G(\omega) = 0$ and the larger of the two positive roots of $F(\omega) = 0$, while the post-shock ω, ω^1, lies between the corresponding smaller roots. With knowledge of the approximate location of the roots, it is a simple matter to solve equation 3.27 numerically (see below).

Weak Shocks - Signal Speed

A weak shock can be defined as one in which the state variables change by only a small amount across the shock. In the limit, when the changes in the state variables can be treated as infinitesmals, the shock becomes a sound wave, i.e., a travelling small disturbance in the fluid. The velocity of weak shocks is thus, in this limit, the signal speed in the fluid.

Define the state variables in the pre-shock state by subscript 0 and those in the post-shock by subscript 1. Assume that:

$$T_1 = T_o(1 + t) \tag{3.29}$$

and

$$V_1 = V_o(1 - v) \tag{3.30}$$

where t and v are positive numbers, both $\ll 1$. Then, since $\rho_o V_o = \rho_1 V_1$ and $V_o H_o = V_1 H_1$;

$$\rho_1 = \rho_o(1 + v) \tag{3:31}$$

$$H_1 = H_o(1 + v) \tag{3:32}$$

At states 0 and 1 (where all derivatives vanish):

$$P = mV + \rho RT + \frac{1}{3} aT^4 + \frac{\mu H^2}{2} \tag{3.33}$$

and

$$Q = \frac{mV^2}{2} + \frac{\gamma}{\gamma - 1} mRT + \frac{4}{3} VaT^4 + \mu VH^2 \tag{3.34}$$

If equations 3.33 and 3.34 are written for states 0 and 1, and equations 3.29 to 3.32 are substituted, the results are:

$$(t + v)\rho_o RT_o + \frac{4}{3} taT_o^4 - mV_o v + \mu H_o^2 v = 0 \tag{3:35}$$

$$-mV_o^2 v + \frac{\gamma}{\gamma - 1} mRT_o t + (4t - v)\frac{4}{3} aV_o T_o^4 + \mu V_o H_o^2 v = 0 \tag{3:36}$$

Equations e.35 and 3:36 can be re-written as:

$$(1 + \frac{4}{3} \epsilon)t + (1 - \gamma')v = 0 \tag{3:37}$$

$$\left(\frac{\gamma}{\gamma - 1} + \frac{16}{3}\,\epsilon\right)t - \left(\gamma' + \frac{4}{3}\,\epsilon\right)v = 0 \qquad (3.38)$$

where:

$$\gamma' = \frac{V_o^2 - \frac{\mu H_o^2}{\rho_o}}{RT_o} \qquad (3.39)$$

$$\epsilon = \frac{aT_o^3}{\rho_o R} \qquad (3.40)$$

Equations 3.37 and 3.38 are two simultaneous, homogeneous linear equations, so that for t and v to have non-zero solutions, the determinant of their coefficients should vanish. This gives:

$$\gamma' = 1 + \frac{(1 + \frac{4}{3}\,\epsilon)^2}{(\frac{1}{\gamma - 1} + 4\epsilon)} \qquad (3.41)$$

and

$$t = \frac{1 + \frac{4}{3}\,\epsilon}{\frac{1}{\gamma - 1} + 4\epsilon}\,v \qquad (3.42)$$

so that:

$$V_o^2 = RT_o\left(1 + \frac{(1 + \frac{4}{3}\,\epsilon)^2}{\frac{1}{\gamma - 1} + 4\epsilon}\right) + \frac{\mu H_o^2}{\rho_o}$$

The signal speed is:

$$V_o = \sqrt{\gamma' RT_o + \mu H_o^2/\rho_o} \tag{3.43}$$

where γ' is given by equation 3.41.

This value for signal speed can also be found by considering the jump equations as state equations and using the results of MHD for the signal speed (see Appendix B).

The "Mach Number" for radiative shocks will be defined as the ratio of the velocity to the signal speed:

$$M = \frac{V}{\sqrt{\gamma' RT + \mu H^2/\rho}} \tag{3.44}$$

in terms of the dimensionless variables,

$$\frac{1}{M^2} = \frac{\gamma'\theta}{\omega^2} + \frac{1}{\omega}\frac{\chi^2}{Z} \tag{3.45}$$

Since the weak shock solution corresponds to $M_o = M_1 = 1$, states which satisfy equation 3.27 with finite changes in fluid properties will have

$$\begin{aligned} M_o &> 1 \\ M_1 &< 1 \end{aligned} \tag{3.46}$$

(see Fig. 3.2).

Strong Shocks

When the shock propagates at large Mach Number into a gas at fairly low temperature, approximate results for the post-shock state can be written in terms of the parameters of the pre-shock state. Let

$$\delta = \frac{RT_o}{v_o^2} \tag{3.47}$$

$$\epsilon = \epsilon_o = \frac{aT_o^3}{\rho_o R} \tag{3.48}$$

$$\chi = \chi_o = \frac{\mu H_o^2}{\eta_o v_o^2} \tag{3.49}$$

where it is assumed that δ, ϵ, an χ are all much less than 1 (in which case χ, as defined in equation 3.49, is the same as the χ used previously). This assumption means that not only is the kinetic energy of the fluid much larger than the thermal or magnetic energy present, but the initial conditions are such that the radiative energy is much less than the thermal energy.

Then:

$$m = \rho_o v_o \tag{3.50}$$

$$P = mv_o[1 + \delta + \frac{1}{2}\chi] \tag{3.51}$$

$$Q = mv_o^2[\frac{1}{2} + \frac{\gamma}{\gamma-1}\delta + \chi] \tag{3.52}$$

and, to first order in ϵ, δ and χ:

$$A = \frac{2Qm}{P^2} - 1 = \frac{\delta}{D} + \chi \tag{3.53}$$

$$\omega^o = \frac{mv_o}{P} = 1 - \delta - \frac{1}{2}\chi \tag{3.54}$$

$$\theta^o = \frac{m^2RT_o}{P^2} = \delta \tag{3.55}$$

$$S = \frac{aP^7}{m^8R^4} = \frac{\epsilon}{\delta^3} \tag{3.56}$$

$$Z = \frac{E^2m^2}{\mu p^3} = \chi \tag{3.57}$$

The jump equation, 3.27, can now be solved for ω^1, assuming that $\omega^1 = \frac{1}{7} + \alpha$ where α is a lower order infinitesimal than δ, ϵ or χ.

To lowest order, equation 3.27 becomes:

$$\frac{(\frac{1}{7})^3(\frac{6}{7})(\frac{7-q}{7})}{(\frac{6}{7})^4\alpha^4} = \frac{\epsilon}{3\delta^3f^3} \tag{3.58}$$

or

$$\alpha = \frac{1}{6}\left(\frac{18}{7}\right)^{1/4}\left(\frac{\delta^3}{e}\right)^{1/4} f \tag{3.59}$$

$$\omega^1 = \frac{1}{7}[1 + (3)^{1/4}(\frac{7}{6})^{3/4}(\frac{\delta^3}{\epsilon})^{1/4} f] \tag{3.60}$$

Equation 3.59 is consistent with the assumption that α is a lower order infinitesimal.

Solving equation 3.12,

$$\theta^1 = \left(\frac{18}{7}\right)^{1/4}\left(\frac{\delta^3}{\epsilon}\right)^{1/4} \tag{3.61}$$

Also:

$$\epsilon_1 = S\omega^1\theta^{1^3} = \frac{1}{7}\left(\frac{18}{7}\right)^{3/4}\left(\frac{\epsilon}{\delta^3}\right)^{1/4} \tag{3.62}$$

$$\frac{\rho_1}{\rho_o} = \frac{\chi_1}{\chi} = \frac{\omega^o}{\omega^1} = 7\left[1 - 3^{1/4}\left(\frac{7}{6}\right)^{3/4}\left(\frac{\delta^3}{\epsilon}\right)^{1/4} f\right] \tag{3.63}$$

$$\frac{P_1}{P_o} = \frac{\rho_1\theta^1}{\rho_o\theta^o} = 7\left(\frac{18}{7}\right)^{1/4}\frac{1}{(\epsilon\delta)^{1/4}} \tag{3.64}$$

$$\frac{T_1}{T_o} = \frac{\theta^1}{\theta^o} = \left(\frac{18}{7}\right)^{1/4}\frac{1}{(\epsilon\delta)^{1/4}} \tag{3.65}$$

$$\frac{1}{M_1^2} = \frac{\gamma_1'\theta^1}{\omega^{1^2}} + \frac{1}{\omega^1}\frac{\chi_1^2}{Z} = 8\left[1 - 2(3)^{1/4}\left(\frac{7}{6}\right)^{3/4}\left(\frac{\delta^3}{\epsilon}\right)^{1/4} f\right]$$

or

$$M_1 = \frac{1}{\sqrt{8}}\left[1 + 3^{1/4}\left(\frac{7}{6}\right)^{3/4}\left(\frac{\delta^3}{\epsilon}\right)^{1/4} f\right] \tag{3.66}$$

all the above being given to lowest order. Note that the limiting density ratio (7) and the limiting Mach number $(\frac{1}{\sqrt{8}})$

for finite ϵ as $\delta \rightarrow 0$ are the same as for gas dynamic shocks with $\gamma = 4/3$.

The pressure ratio can be expressed in terms of the density ratio $X = \frac{\rho_1}{\rho_o}$ as:

$$\frac{p_1}{p_o} = \frac{(7f)^{1/3}}{(\frac{1}{3}\epsilon)^{1/3}} \frac{X}{(7-X)^{1/3}} \tag{3.67}$$

This agrees with the result of Sachs (Ref. 21):

$$\frac{p_1}{p_o} = \frac{X^{4/3}}{(\frac{1}{3}\epsilon)^{1/3}} \left(\frac{X-g}{7-X}\right)^{1/3} \tag{3.68}$$

to lowest order, since $X = 7 + \mathcal{O}(\alpha)$ and $f = (7-g) = X-g + \mathcal{O}(\alpha)$.

Relationships Among the Constants - Inaccessible States

Equation 3.27 depends on three parameters (A, Z and S) for a given gas. Specification of these three parameters will give a unique pre-shock state and a unique post-shock state. However, physically, the usual case is for the pre-shock state to be given and the post-shock state to be unknown. It is convenient to specify the pre-shock state in terms of three parameters:

M_o, the Mach number

ϵ_o, the ratio of radiation to thermal energy, and

ϵ_{ho}, the ratio of magnetic to thermal energy

$$M_o^2 = \frac{v_o^2}{\gamma_o' RT_o + \frac{\mu H_o^2}{\rho_o}} = \frac{v_o^2}{RT_o(\gamma_o' + \epsilon_{ho})} \tag{3.69}$$

$$\epsilon_o = \frac{aT_o^3}{\rho_o R} \tag{3.70}$$

$$\epsilon_{ho} = \frac{\mu H_o^2}{\rho_o RT_o} \tag{3.71}$$

Using the defining equations for A, Z and S:

$$A = \frac{2Qm}{P^2} - 1$$

$$Z = \frac{E^2 m^2}{\mu P^3}$$

$$S = \frac{aP^7}{m^8 R^4}$$

along with equations 3.33 and 3.34, and the equilibrium relation $E = \mu v_o H_o$, the following expressions are obtained for the parameters:

$$A = \frac{2(\gamma_o' + \epsilon_{ho})[\frac{1}{\gamma-1} + \epsilon_o + \frac{1}{2}\epsilon_{ho}]M_o^2 - [1 + \frac{1}{3}\epsilon_o + \frac{1}{2}\epsilon_{ho}]^2}{[1 + \frac{1}{3}\epsilon_o + \frac{1}{2}\epsilon_{ho} + (\gamma_o' + \epsilon_{ho})M_o^2]^2} \tag{3.72}$$

$$Z = \frac{\epsilon_{ho}[\gamma_o' + \epsilon_{ho}]^2 M_o^4}{[1 + \frac{1}{3}\epsilon_o + \frac{1}{2}\epsilon_{ho} + (\gamma_o' + \epsilon_{ho})M_o^2]^3} \tag{3.73}$$

$$S = \frac{\epsilon_o[1 + \frac{1}{3}\epsilon_o + \frac{1}{2}\epsilon_{ho} + (\gamma_o' + \epsilon_{ho})M_o^2]^7}{(\gamma_o' + \epsilon_{ho})^4 M_o^8} \tag{3.74}$$

For $M_o \to \infty$, these reduce to:

$$A = \frac{2[\frac{1}{\gamma-1} + 2\epsilon_o + \epsilon_{ho}]}{(\gamma_o' + \epsilon_{ho})M_o^2} \tag{3.75}$$

$$Z = \frac{\epsilon_{ho}}{(\gamma_o' + \epsilon_{ho})M_o^2} \tag{3.76}$$

$$S = \epsilon_o(\gamma_o' + \epsilon_{ho})^3 M_o^6 \tag{3.77}$$

S gets very large as M_o increases, provided $\epsilon_o \neq 0$.

For $\epsilon_o = \epsilon_{ho} = 0$, $Z = S = 0$, and A assumes its value for an ordinary gas-dynamic shock:

$$A = \frac{\frac{2\gamma}{\gamma-1} M_o^2 - 1}{(1 + \gamma M_o^2)^2} \quad \text{(Ref. 41)}$$

Relations 3.72 and 3.74 apply to the final state as well, and can be used to define states inaccessible from shocks.

From Fig. 3.1 it is clear that for $A < -hZ$ there exists no pre-shock state, since the line with slope $-\tan^{-1}A$ and intercept $-hZ$ will intersect the cubic curves at points $\omega^o > 1$. Therefore, the condition that a "post-shock" state 1 be accessible from a pre-shock state is that

$$A \geq -hZ$$

or

$$\frac{h\epsilon_{h1}[\gamma_1' + \epsilon_{h1}]M_1^4}{[1 + \frac{1}{3}\epsilon_1 + \frac{1}{2}\epsilon_{h1} + (\gamma_1' + \epsilon_{h1})M_1^2]^2} + 2(\gamma_1' + \epsilon_{h1})[\frac{1}{\gamma-1} + \epsilon_1 + \frac{1}{2}\epsilon_{h1}]M_1^2$$

$$\geq [1 + \frac{1}{3}\epsilon_1 + \frac{1}{2}\epsilon_{h1}] \qquad (3.78)$$

which gives a relation among the three parameters M_1, ϵ_1 and ϵ_{h1}, which must be satisfied for all accessible states. When $\epsilon_1 = \epsilon_{h1} = 0$, the relation reduces to the familiar "strong shock" solution to the gas dynamic jump equation:

$$M_1^2 \geq \frac{\gamma - 1}{2\gamma} \qquad (3.79)$$

The equality corresponding to relation 3.78 can be treated as a cubic equation in M_1^2:

$$a_3M_1^6 + a_2M_1^4 + a_1M_1^2 - a_o = 0 \tag{3.80}$$

with all a's greater than zero (for $\gamma \leq 2$).

By Descartes' rule of signs, there is a real positive root of 3.80, Mm^2, such that, for given ϵ_o and ϵ_{ho}, all Mach numbers $1 \geq M \geq Mm$ are accessible from pre-shock states.

Numerical Results

Equation 3.27 has been solved numerically for ionized hydrogen with H = 0 and the following ranges of variables:

$$2 \leq M_o \leq 50$$

$$0.1 \leq \epsilon_o \leq 10 \;;$$

also $\epsilon_o = 0$ (radiation ignored)

and the results have been plotted in Figs. 3.2 to 3.7. Figure 3.2 is a plot of M_1 vs M_o, and it is to be noted that for $\epsilon_o \neq 0$, all curves approach $M_1 = \frac{1}{\sqrt{8}}$ as M_o increases, which satisfies equation 3.79 with $\gamma = 4/3$, while for $\epsilon_o = 0$, M_1 approaches $1/\sqrt{5}$, which satisfies equation 3.79 with $\gamma = 5/3$. Figure 3.3 is a plot of $\rho_1/\rho_o(= v_o/v_1)$ vs M_o. Again, the limiting value of density ratio with finite ϵ is 7, which is $g(4/3)$, while, when radiation is ignored, the limit is 4, which is $g(5/3)$.

Figure 3.4 is a plot of temperature ratio vs M_o, from which it is seen that consideration of radiation considerably lowers the theoretical temperature ratio across the shock. That the theoretical pressure ratio is also lowered when radiation is taken into account is demonstrated by Fig. 3.5. However, Fig. 3.6 shows that the total (thermodynamic and radiation) pressure ratio across the shock is largely independent of ϵ_o, and is approximately equal to the thermodynamic pressure ratio across a non-radiative shock. Figure 3.7 is a plot of total pressure ratio divided by thermodynamic pressure ratio, i.e., $(1 + \frac{1}{3}\epsilon_1)/(1 + \frac{1}{3}\epsilon_o)$ vs M_o, showing the importance of radiation at Mach numbers above about 5, even when the initial value of ϵ is small.

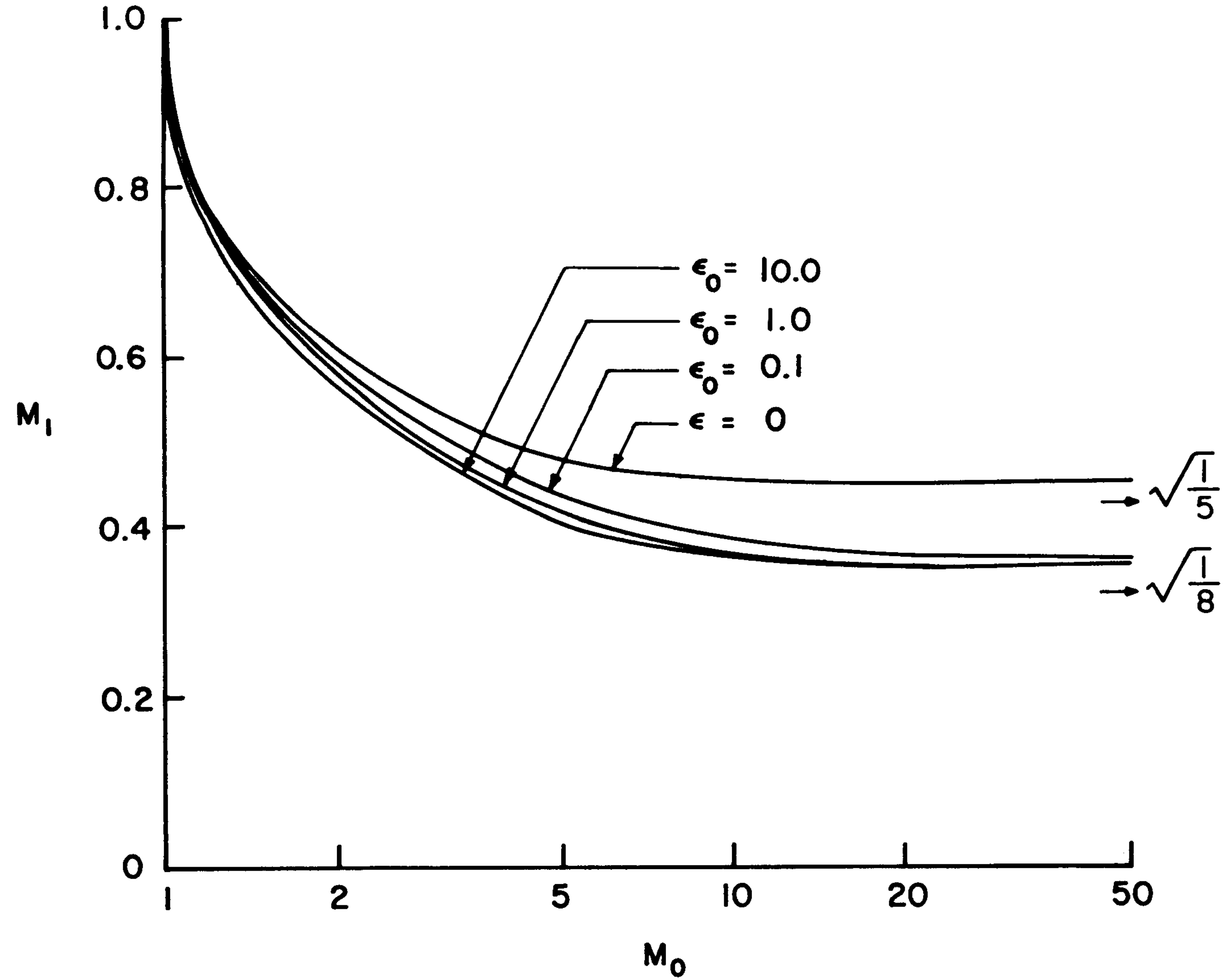

FIG. 3.2 M_1 vs. M_0

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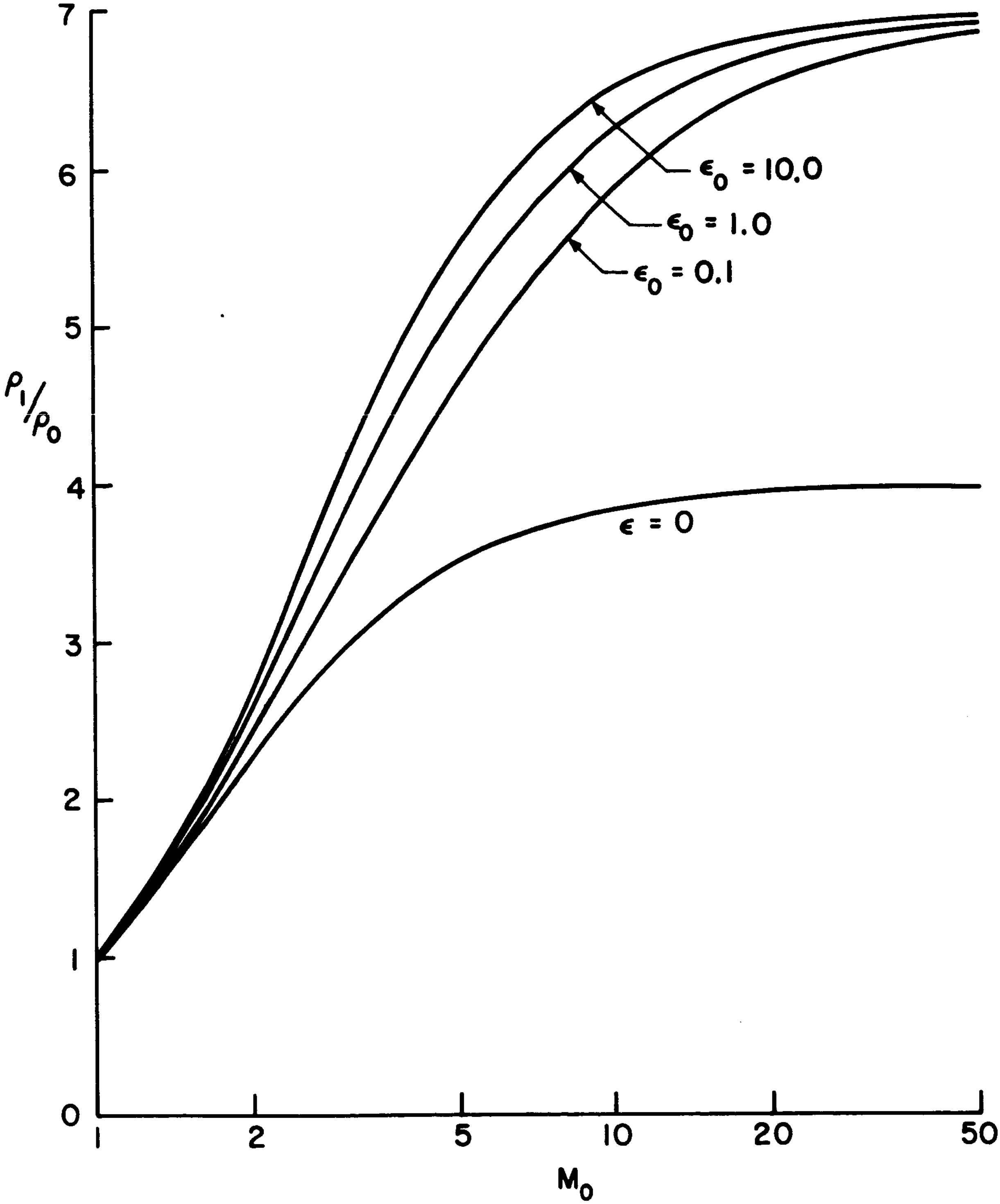

FIG. 3.3 $\frac{\rho_1}{\rho_0}$ vs. M_0

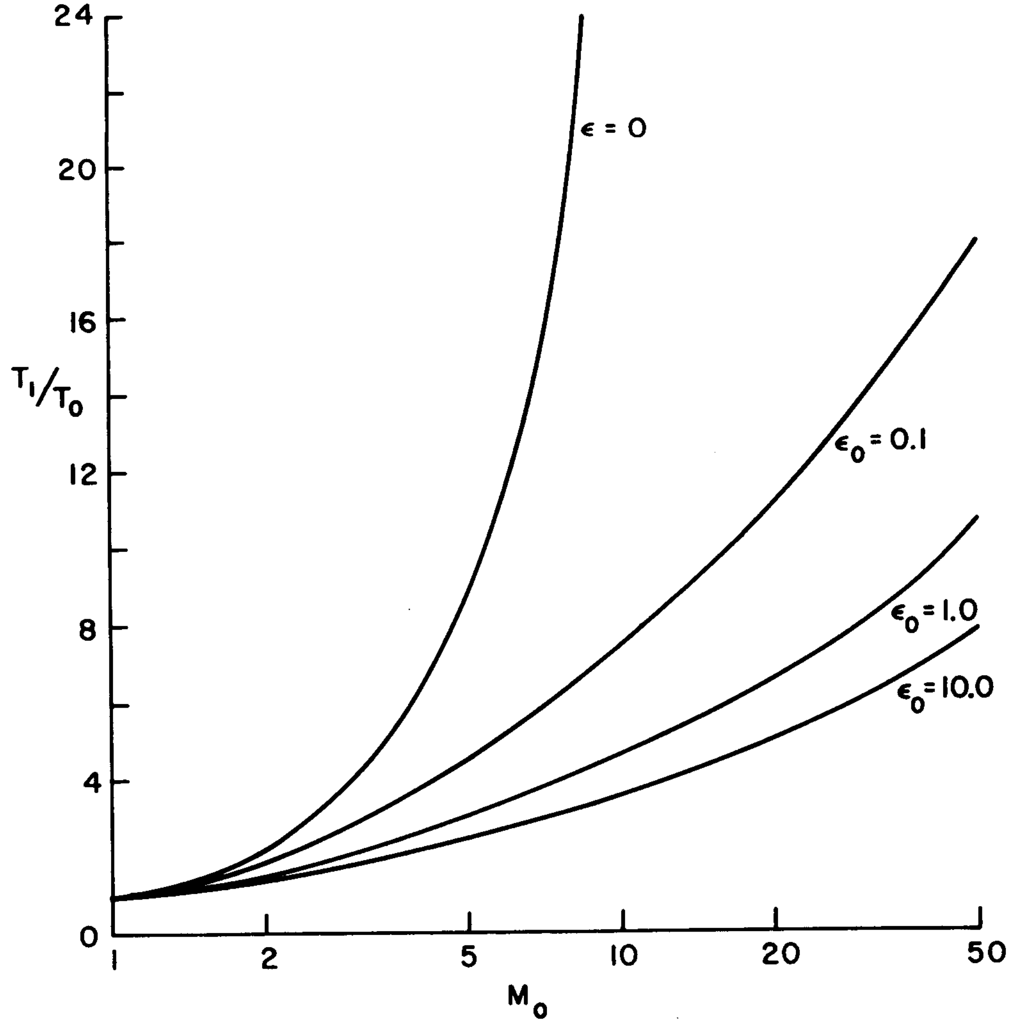

FIG. 3.4 $\frac{T_1}{T_0}$ vs. M_0

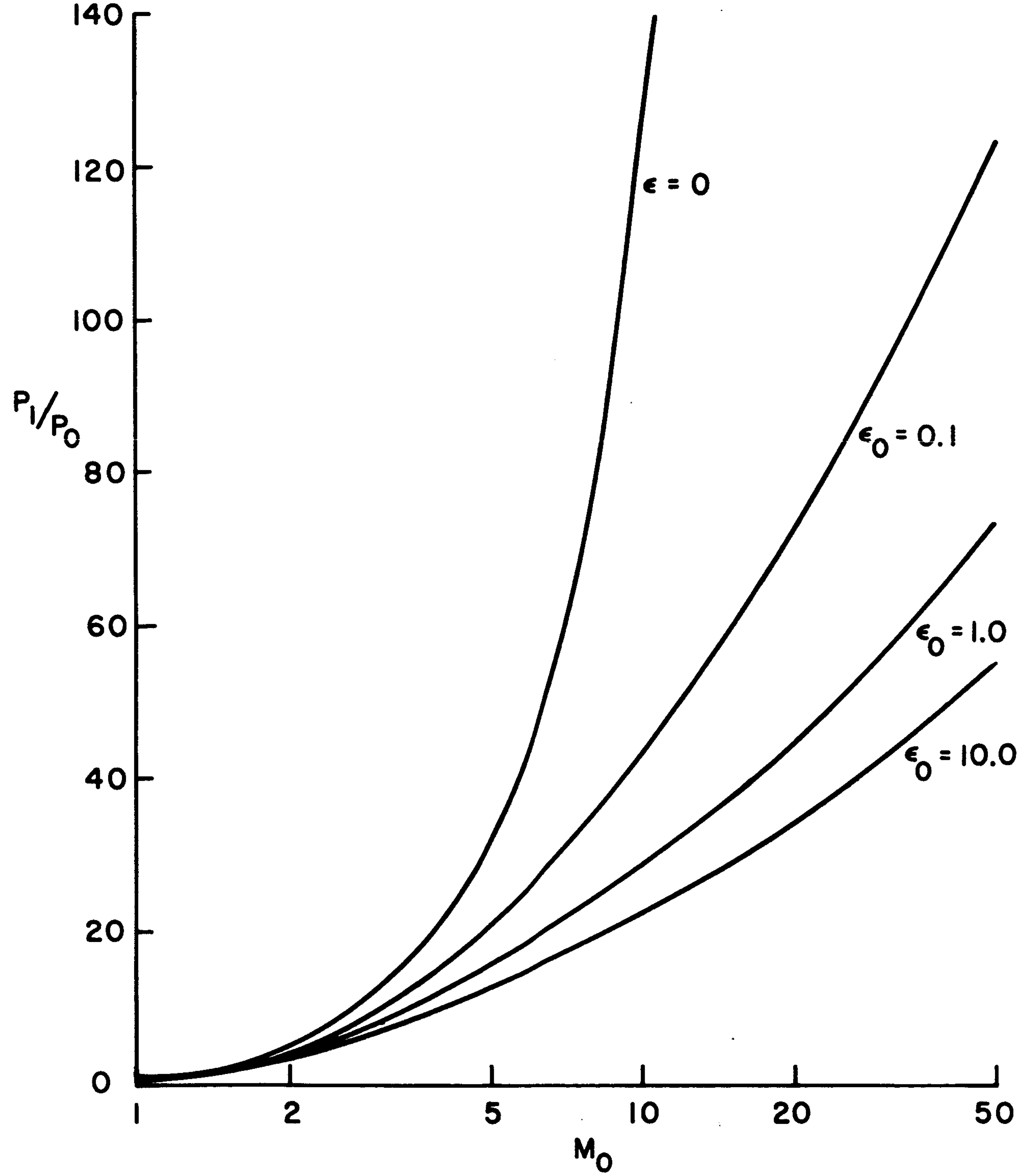

FIG. 3.5 $\frac{P_1}{P_0}$ vs. M_0

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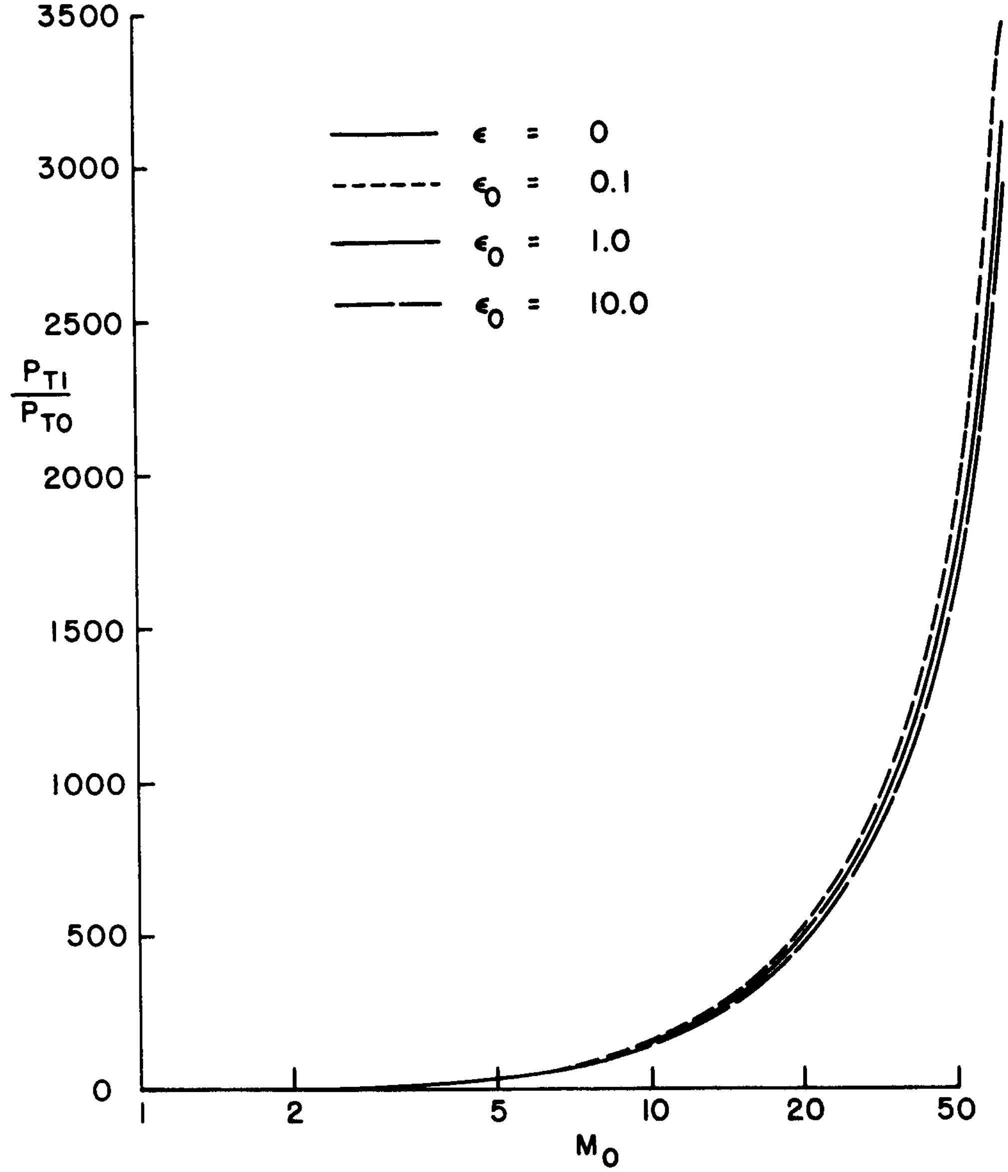


FIG. 3.6 $\frac{P_{T1}}{P_{T0}}$ (TOTAL PRESSURE RATIO) vs. M_0

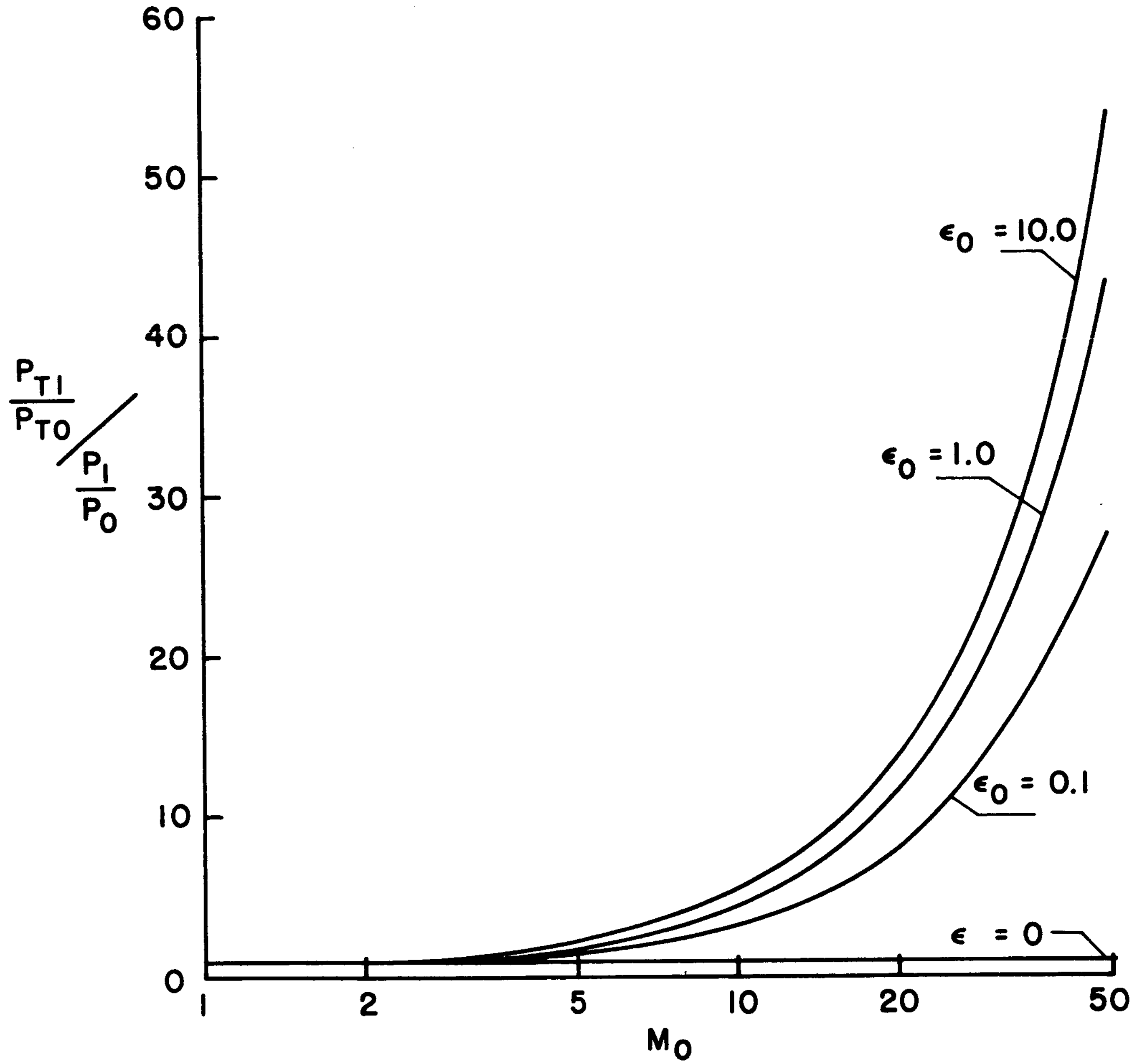


FIG 3.7 $\frac{P_{T1}}{P_{T0}} \Big/ \frac{P_1}{P_0}$ vs. M_0

CHAPTER IV

THE MOMENT APPROXIMATION OF THE STRUCTURE EQUATIONS

The shock structure equations of Chapter II, repeated here:

$$L_P \frac{d\omega}{dx} = \frac{\theta}{\omega} + \omega + p_r' + \frac{1}{2}\frac{\chi^2}{Z} - 1 \tag{4.1}$$

$$L_H \frac{d\theta}{dx} = \theta + 2D\omega U_r' - D[(1-\omega)^2 + A] + D\chi[2 - \frac{\chi\omega}{Z}] + \frac{2D}{\beta} F' \tag{4.2}$$

$$L_M \frac{d\chi}{dx} = \omega\chi - Z \tag{4.3}$$

$$L_{R\nu}\ell \frac{\partial\phi_\nu}{\partial x} = B_\nu'(\theta) - \phi_\nu \tag{4.4}$$

$$F' = \frac{1}{2}\int_{-1}^{1} \ell d\ell \int_{0}^{\infty} d\nu\phi_\nu \tag{4.5}$$

$$U_r' = \frac{1}{2}\int_{-1}^{1} d\ell \int_{0}^{\infty} d\nu\phi_\nu \tag{4.6}$$

$$p_r' = \frac{1}{2}\int_{-1}^{1} \ell^2 d\ell \int_{0}^{\infty} d\nu\phi_\nu \tag{4.7}$$

are a set of integro-differential equations in three independent variables, x, ℓ and ν, the solution of which is quite difficult. The set can be changed into an infinite set of ordinary differential equations with x the dependent variable and ν a parameter, which, through a mathematical approximation, called the "moment approximation" in radiative transfer theory, can be reduced to a finite set (Ref. 42).

The change to ordinary differential equations comes from the expansion of ϕ_ν in a Fourier series in Legendre polynomials:

$$\phi_\nu(x, \ell) = \sum_{n=o}^{\infty} \phi_{\nu n}(x) P_n(\ell) \tag{4.8}$$

where

$$P_n(\ell) = \frac{1}{2^n n!} \frac{d^n}{d\ell^n}(\ell^2 - 1)^n \tag{4.9}$$

is the n'th Legendre polynomial in ℓ. The formula for the n'th coefficient $\phi_{\nu n}(x)$ is:

$$\phi_{\nu n}(x) = \frac{2n+1}{2} \int_{-1}^{1} \phi_\nu(x, \ell) P_n(\ell) d\ell \tag{4.10}$$

From the definition 4.9 it is clear that:

$$1 = P_o(\ell) \tag{4.11}$$

$$\ell = P_1(\ell) \tag{4.12}$$

$$\ell^2 = \frac{1}{3} P_o(\ell) + \frac{2}{3} P_2(\ell) \tag{4.13}$$

and, it is also true that

$$\int_{-1}^{1} P_n(\ell) P_m(\ell) d\ell = \frac{2}{2n+1} \delta_{nm}$$

where δ_{nm} is the Kronecker delta, so that:

$$F' = \frac{1}{3} \int_{o}^{\infty} \phi_{\nu 1} d\nu \tag{4.14}$$

$$U_r' = \int_{o}^{\infty} \phi_{\nu o} d\nu \tag{4.15}$$

$$p_r' = \frac{1}{3} \int_{o}^{\infty} \phi_{\nu o} d\nu + \frac{2}{15} \int_{o}^{\infty} \phi_{\nu 2} d\nu \tag{4.16}$$

Equation 4.4 can be changed from a single partial differential equation to an infinite number of ordinary differential equations by multiplying both sides by each of the Legendre polynomials in turn and integrating over ℓ. Equation 4.4 can first be rewritten as:

$$L_{R\nu} \ell \sum_{n=o}^{\infty} \frac{d\phi_{\nu n}(x)}{dx} P_n(\ell) = B_\nu'(\theta) - \sum_{n=o}^{\infty} \phi_{\nu n}(x) P_n(\ell) \tag{4.17}$$

If equation 4.17 is integrated over ℓ from -1 to 1, the result is:

$$\frac{1}{3} L_{R\nu} \frac{d\phi_{\nu 1}}{dx} = B_\nu'(\theta) - \phi_{\nu o} \tag{4.18}$$

which is the zero'th moment of equation 4.17. For the first and succeeding moments it is convenient to use the recurrence relationship:

$$\ell P_n(\ell) = \frac{n+1}{2n+1} P_{n+1}(\ell) + \frac{n}{2n+1} P_{n-1}(\ell) \tag{4.19}$$

so that equation 4.17 becomes

$$L_{R\nu}\left[\sum_{n=1}^{\infty} \frac{d\phi_{\nu n-1}}{dx} \frac{n}{2n-1} P_n(\ell) + \sum_{n=o}^{\infty} \frac{d\phi_{\nu n+1}}{dx} \frac{n+1}{2n+3} P_n(\ell)\right]$$

$$= B_\nu'(\theta) - \sum_{n=o}^{\infty} \phi_{\nu n}(x) P_n(\ell) \tag{4.20}$$

The zero'th moment of equation 4.20 is equation 4.18. The first moment is:

$$L_{R\nu}\left[\frac{d\phi_{\nu o}}{dx} + \frac{2}{5}\frac{d\phi_{\nu 2}}{dx}\right] = -\phi_{\nu 1} \tag{4.21}$$

The second moment is:

$$L_{R\nu}\left[\frac{2}{3}\frac{d\phi_{\nu 1}}{dx} + \frac{3}{7}\frac{d\phi_{\nu 3}}{dx}\right] = -\phi_{\nu 2} \tag{4.22}$$

The m'th moment $(m \neq 0)$ is:

$$L_{R\nu}\left[\frac{m}{2m-1}\frac{d\phi_{\nu m-1}}{dx} + \frac{m+1}{2m+3}\frac{d\phi_{\nu m+1}}{dx}\right] = -\phi_{\nu m} \tag{4.23}$$

Equation 4.23 can be solved directly for $\frac{d\phi_{\nu m+1}}{dx}$ when m is an even number, using equation 4.18. For instance, from equations 4.18 and 4.22:

$$\frac{3}{7} L_{R\nu} \frac{d\phi_{\nu 3}}{dx} = - \phi_{\nu 2} + 2(\phi_{\nu o} - B_{\nu}'(\theta)) \tag{4.24}$$

In general, from equation 4.23:

$$L_{R\nu} \frac{m+1}{2m+3} \frac{d\phi_{\nu m+1}}{dx} = - \phi_{\nu m} - \frac{m}{2m-1} L_{R\nu} \frac{d\phi_{\nu m-1}}{dx} \tag{4.25}$$

if $m - 2$ is substituted for m in equation 4.25, the result is:

$$L_{R\nu} \frac{m-1}{2m-1} \frac{d\phi_{\nu m-1}}{dx} = - \phi_{\nu m-2} - \frac{m-2}{2m-1} L_{R\nu} \frac{d\phi_{\nu m-3}}{dx} \tag{4.26}$$

and 4.25 and 4.26 can be combined into:

$$L_{R\nu} \frac{m+1}{2m+3} \frac{d\phi_{\nu m+1}}{dx} = - \phi_{\nu m} + \frac{m}{m-1} \phi_{\nu m-2} + \frac{m}{m-1} \frac{m-2}{2m-5} L_{R\nu} \frac{d\phi_{\nu m-3}}{dx} \tag{4.27}$$

If this kind of substitution is carried out $\frac{1}{2}$ m times, and equation 4.18 is used in the last substitution, the result is:

$$L_{R\nu} \frac{m+1}{2m+3} \frac{d\phi_{\nu m+1}}{dx} = - \phi_{\nu m} - \sum_{k=1}^{m/2} B_k{}^m(\phi_{\nu m-2k} - \delta_{o,m-2k} B_{\nu}'(\theta)) \tag{4.28}$$

where

$$B_k^m = (-1)^k \prod_{j=o}^{k-1} \frac{m-2j}{m-2j-1} \quad (4.29)$$

Thus, $\phi_{\nu m}$ is determined, for m odd, by a differential equation involving all the even $\phi_{\nu m}$'s with $n < m$, and θ.

Equation 4.23 cannot be solved directly when m is odd, since $\phi_{\nu m-1}$ is only given in terms of $\phi_{\nu m+1}$ and $\phi_{\nu m}$ and no equation exists for $\phi_{\nu o}$ alone (equation 4.21 can be solved for $\phi_{\nu o}$ only in terms of $\phi_{\nu 2}$). Thus, the set of equations 4.23 is an infinite set, with each new moment equation introducing a new variable.

Sets of equations similar to 4.23 occur very frequently in radiative transfer theory. A common way in which the infinite set is terminated is by assuming arbitrarily that $\phi_{\nu 2N} \equiv 0$. The resulting equations are called the "N'th moment approximation" (Ref. 42). It can be shown (Ref. 42) that the N'th moment approximation is mathematically equivalent to the N'th Gaussian quadrature of Chandrasekhar (Ref. 36). The assumption that $\phi_{\nu 2N} \equiv 0$ is thus a mathematical one, with the accuracy in the numerical determination of ϕ_ν:

$$\phi_\nu = \sum_{n=o}^{2N-1} \phi_{\nu n}(x) P_n(\ell) \quad (4.30)$$

increasing as N increases. It is to be noted that if $\phi_{\nu 2N} \equiv 0$, $\phi_{\nu 2N+1} \neq 0$ because of equation 4.23, and all $\phi_{\nu n}$'s of order $n > 2N$ cannot be identically zero. However, there now exists a complete set of equations consisting of equation 4.28 for m odd, and another group of equations, to be determined below, for m even.

Taking equation 4.23 for $m = 2N - 1$ results in:

$$L_{R\nu} \frac{2N-1}{4N-3} \frac{d\phi_{\nu 2N-2}}{dx} = - \phi_{\nu 2N-1} \tag{4.31}$$

and, for $M = 2N - 3$:

$$L_{R\nu} \frac{2N-3}{4N-7} \frac{d\phi_{\nu 2N-4}}{dx} + \frac{2N-2}{4N-3} \frac{d\phi_{\nu 2N-2}}{dx} = - \phi_{\nu 2N-3} \tag{4.32}$$

Substituting equation 4.31 in equation 4.32:

$$\frac{2N-3}{4N-7} L_{R\nu} \frac{d\phi_{\nu 2N-4}}{dx} = - \phi_{\nu 2N-3} + \frac{2N-2}{2N-1} \phi_{\nu 2N-1} \tag{4.33}$$

For m any even number, if the above substitution is carried out $\frac{2N-m}{2} - 1$ times, the result is:

$$\frac{m+1}{2m+1} L_{R\nu} \frac{d\phi_{\nu m}}{dx} = - \phi_{\nu m+1} - \sum_{k=1}^{N-1-\frac{m}{2}} C_k^m \phi_{\nu m+2k+1} \tag{4.34}$$

where

$$C_k^m = (-1)^k \prod_{j=1}^{k} \frac{m+2j}{m+2j+1} \tag{4.35}$$

Thus $\phi_{\nu m}$ is determined, for m even, by a differential equation involving all the odd $\phi_{\nu n}$'s with $2N - 1 \geq n \geq m$.

Equation 4.1 and 4.2 can be rewritten, in view of equations 4.14 to 4.16, as:

$$L_P \frac{d\omega}{dx} = \frac{\theta}{\omega} + \omega + \frac{1}{3}\phi_o + \frac{2}{15}\phi_2 + \frac{1}{2}\frac{\chi^2}{Z} - 1 \tag{4.36}$$

$$L_H \frac{d\theta}{dx} = \theta + 2D\omega\phi_o - D[(1-\omega)^2 + A] + D\chi(2 - \frac{\chi\omega}{Z}) + \frac{2D}{3\beta}\phi_1 \tag{4.37}$$

where

$$\phi_o = \int_o^\infty \phi_{\nu o} d\nu$$

$$\phi_1 = \int_o^\infty \phi_{\nu 1} d\nu$$

$$\phi_2 = \int_o^\infty \phi_{\nu 2} d\nu$$

Equations 4.28 and 4.34 represent a set of 2N ordinary differential equations with ν as a parameter. Equations 4.36 and 4.37, on the other hand, have the ν dependence removed. In general, in order for the moment approximation equations 4.28 and 4.34 to be of value, their ν dependence must also be removed. This can be done by an appropriate definition of a radiative (frequency) mean free path, e.g., the Rosseland Mean Free Path used in the diffusion approximation (see Chapter V). Assuming that the mean free path L_R has been determined, equations 4.28 and 4.34 become respectively:

$$L_R \frac{m+1}{2m+3} \frac{d\phi_{m+1}}{dx} = - \phi_m + \sum_{k=1}^{m/2} B_k^m(\phi_{m-2k} - \delta_{o,m-2k} S\theta^4) \qquad (4.38)$$

$$L_R \frac{m+1}{2m+1} \frac{d\phi_m}{dx} = - \phi_{m+1} - \sum_{k=1}^{N-1-\frac{m}{2}} C_k^m \phi_{m+2k+1} \qquad (4.39)$$

where m is even, B_k^m and C_k^m are defined as before and

$$\phi_m = \int_o^\infty \phi_{\nu m} d\nu \qquad (4.40)$$

The set of structure equations now consists, in the N'th moment approximation, of the following 2N + 3 equations: 4.36, 4.37, 4.3, the N equations represented by 4.38, and the N equations represented by 4.39. There are 2N + 3 unknowns: ω, θ, χ and the 2N variables $\phi_o \ldots \phi_{2N-1}$. In particular, the equations for the first moment approximation are:

$$L_P \frac{d\omega}{dx} = \frac{\theta}{\omega} + \omega + \frac{1}{3}\phi_o + \frac{1}{2}\frac{\chi^2}{Z} - 1 \qquad (4.41)$$

$$L_H \frac{d\theta}{dx} = \theta + 2\mathbf{D}\omega\phi_o - \mathbf{D}[(1-\omega)^2 + A] + \mathbf{D}\chi[2 - \frac{\chi\omega}{Z}] + \frac{2}{3}\frac{\mathbf{D}}{\beta}\phi_1 \qquad (4.42)$$

$$L_M \frac{d\chi}{dx} = \omega\chi - Z \tag{4.43}$$

$$L_R \frac{d\phi_o}{dx} = - \phi_1 \tag{4.44}$$

$$\frac{1}{3} L_R \frac{d\phi_1}{dx} = S\theta^4 - \phi_o \tag{4.45}$$

It will be assumed henceforth that equations 4.41 - 4.45 are the correct structure equations. The physical assumption introduced by consideration of the first moment approximation alone is that the radiative intensity at any point in this plane-stratified atmosphere can be approximated by the sum of two terms, one isotropic and one representing forward (or backward) radiation. For this problem, where the atmosphere is considerably hotter at $x \to \infty$ than it is at $x \to -\infty$, there will be a tendency for much of the radiative intensity to be in the backward direction so that, since the second Legendre polynomial P_2 has a maximum value in directions perpendicular to the axis equal to 1/2 of its value in the forward and backward directions, the assumption that its Fourier coefficient ϕ_2 is zero is a good one.

The method of solution of equations 4.41 - 4.45 is discussed in Appendix C and in Chapter V.

CHAPTER V

THE SOLUTION OF THE SHOCK STRUCTURE EQUATIONS IN THE DIFFUSION (EDDINGTON) APPROXIMATION

The radiative shock structure problem has now been reduced, in its simplest form, to the five equations 4.41 to 4.45. The solution of these equations is, however, still a formidable problem. A further simplification can be obtained if it is assumed that the radiative mean free path is small compared to the characteristic lengths in which the variables change. This leads to what is known in the literature of radiative transfer as the diffusion or Eddington approximation (Ref. 43).

The equations for $\phi_{\nu 1}$ and $\phi_{\nu o}$ in the first moment approximation are (see Chapter IV):

$$L_{R\nu} \frac{d\phi_{\nu o}}{dx} = - \phi_{\nu 1} \tag{5.1}$$

$$\frac{1}{3} L_{R\nu} \frac{d\phi_{\nu 1}}{dx} = B_\nu'(\theta) - \phi_{\nu o} \tag{5.2}$$

If the derivatives of $L_{R\nu}$ are ignored, equation 5.2 can be differentiated and equation 5.1 can be used to give:

$$\frac{1}{3} L_{R\nu}{}^2 \frac{d^2\phi_{\nu 1}}{dx^2} - \phi_{\nu 1} = L_{R\nu} \frac{d}{dx} B_\nu'(\theta) \tag{5.3}$$

$\phi_{\nu 1}$ can then be expanded in a series of powers in $L_{R\nu}$:

$$\phi_{\nu 1} = \sum_{m=1}^{\infty} L_{R\nu}^{m} \phi_{\nu 1}{}^{(m)} \tag{5.4}$$

provided $L_{R\nu}$ is a "small" parameter. The results of substituting 5.4 in 5.3 and equating the coefficients of equal powers of $L_{R\nu}$ are:

$$\phi_{\nu 1}{}^{(m)} = 0 \quad m \quad \text{even} \tag{5.5}$$

$$\phi_{\nu 1}{}^{(1)} = - \frac{d}{dx} B_\nu'(\theta) \tag{5.6}$$

$$\phi_{\nu 1}{}^{(3)} = - \frac{1}{3} \frac{d^3}{dx^3} B_\nu'(\theta) \tag{5.7}$$

$$\phi_{\nu 1}{}^{(2k+1)} = - \frac{1}{3^k} \frac{d^{2k+1}}{dx^{2k+1}} B_\nu'(\theta) \tag{5.8}$$

for all integers k.

The Eddington approximation consists of taking only the first non-zero term of equation 5.4, i.e.,

$$\phi_{\nu 1} = - L_{R\nu} \frac{d}{dx} B_\nu'(\theta) \tag{5.9}$$

and is thus valid if and only if

$$\left| \frac{1}{3} L_{R\nu}{}^2 \frac{d^3}{dx^3} B_\nu'(\theta) \right| \ll \left| \frac{d}{dx} B_\nu'(\theta) \right| \tag{5.10}$$

or in general

$$\left| \frac{1}{3^k} L_{R\nu}{}^{2k+1} \frac{d^{2k+1}}{dx^{2k+1}} B_\nu'(\theta) \right| \ll \left| \frac{d}{dx} B_\nu'(\theta) \right| \tag{5.11}$$

In other words, the operator $L_{R\nu}{}^2 \frac{d^2}{dx^2}$ must be small in absolute value for all values of frequency and at all points in the shock structure. The conditions for which the Eddington approximation is valid will be discussed below. Assuming its validity for the moment, then, from comparison of equations 5.9 and 5.1,

$$\phi_{\nu o} = B_\nu'(\theta) \tag{5.12}$$

so that, integrating over frequency,

$$\phi_o = S\theta^4 \tag{5.13}$$

and

$$\phi_1 = - L_R \frac{d}{dx} S\theta^4 = - 4L_R S\theta^3 \frac{d\theta}{dx} \tag{5.14}$$

where L_R is the Rosseland mean free path defined by

$$L_R = \frac{\int_0^\infty L_{R\nu} \frac{d}{dx} B_\nu(T)d\nu}{\int_0^\infty \frac{d}{dx} B_\nu(T)d\nu} = \frac{\int_0^\infty L_{R\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu} \tag{5.15}$$

(Ref. 44), since

$$\frac{d}{dx} B_\nu(T) = \frac{\partial B_\nu(T)}{\partial T} \frac{dT}{dx}$$

The equations of the Eddington approximation are thus:

$$L_P \frac{d\omega}{dx} = \frac{\theta}{\omega} + \omega + \frac{1}{3} S\theta^4 + \frac{1}{2} \frac{\chi^2}{Z} - 1 \tag{5.16}$$

$$[L_H + \frac{8}{3} \frac{DL_R}{\beta} S\theta^3] \frac{d\theta}{dx} = \theta + 2D\omega S\theta^4 - D[(1-\omega)^2 + A] + D\chi[2 - \frac{\chi\omega}{Z}] \tag{5.17}$$

$$L_M \frac{d\chi}{dx} = \omega\chi - Z \tag{5.18}$$

For the purpose of simplicity the numerical solution of these equations will be carried out for $\chi = Z = 0$ (no magnetic field) only. In that case, the effects of radiation on a gas dynamic shock will be shown, and the equations are:

$$L_P \frac{d\omega}{dx} = \frac{\theta}{\omega} + \omega + \frac{1}{3} S\theta^4 - 1 \equiv F(\omega, \theta) \tag{5.19}$$

$$L_Q \frac{d\theta}{dx} = \theta + 2D\omega S\theta^4 - D[(1-\omega)^2 + A] \equiv G(\omega, \theta) \tag{5.20}$$

where

$$L_Q = L_H + \frac{8}{3} \frac{DL_R}{\beta} S\theta^3 \tag{5.21}$$

Equations 5.21 and 5.22 can be written as a single equation in the $\omega - \theta$ plane (the "phase plane"):

$$\frac{d\theta}{d\omega} = \delta' \frac{G}{F} \tag{5.22}$$

where $\delta' = \frac{L_P}{L_Q}$, a function of x. (5.23)

Examination of the formulas for the characteristic lengths L_P and L_Q shows that $\delta' \ll 1$.

In Chapter 2, the characteristic length L_P was defined as:

$$L_P = \frac{4}{3} \frac{\eta}{m} \tag{5.24}$$

Marshall (Ref. 45) gives the following formula for the viscosity of ionized hydrogen:

$$\eta = \frac{5\sqrt{m_H}\ (kT)^{5/2}}{4\sqrt{\pi}\ e^4\ \psi} \text{ cgs} \tag{5.25}$$

where:

m_H is the mass of the hydrogen ion

e is the electronic charge, and

ψ is a function given by:

$$\psi = \ln\left[1 + \frac{.547 \times 10^{21}}{n}\left(\frac{T}{10^4}\right)^3\right] \tag{5.26}$$

Numerically:

$$\eta = 0.3823 \times 10^{-14} \frac{T^{5/2}}{\psi} \text{ cgs}$$

so that

$$L_p = \frac{4}{3}\frac{\eta}{m} = 0.5097 \times 10^{-14} \frac{T^{5/2}}{m\psi} \text{ cm} \qquad (5.27)$$

In Ref. 45 the following formula is given for the thermal conductivity of ionized hydrogen:

$$k_c = 0.464\sqrt{2}[\frac{75k}{16\sqrt{\pi}} \frac{(kT)^{5/2}}{\sqrt{m_e}\ e^4\psi} \qquad (5.28)$$

where m_e is the mass of an electron. The "Prandtl number" of gas is defined as

$$N_p = \frac{c_v\eta}{k_c} \qquad (5.29)$$

For ionized hydrogen

$$c_v = \frac{R}{\gamma-1} = \frac{3k}{m_H} \qquad (5.30)$$

so that:

$$N_p = 0.4064\sqrt{m_e/m_H} \sim 0.00945 \qquad (5.31)$$

Now

$$\frac{L_H}{L_P} = \frac{3}{4N_P} = 79.3 \tag{5.32}$$

so that

$$L_H = 0.4042 \times 10^{-12} \frac{T^{5/2}}{m\psi} \text{ cm} \tag{5.33}$$

The function ψ can be expressed in terms of T and v as:

$$\psi = \ln(1 + 0.9082 \times 10^{-15} \frac{vT^3}{m}) \tag{5.34}$$

Finally, in terms of the dimensionless variables ω and θ:

$$L_P = 0.5097 \times 10^{-14} \frac{P^5}{m^6 R^{5/2}} \frac{\theta^{5/2}}{\psi} \text{ cm} \tag{5.35}$$

$$L_H = 0.4042 \times 10^{-12} \frac{P^5}{m^6 R^{5/2}} \frac{\theta^{5/2}}{\psi} \text{ cm} \tag{5.36}$$

$$\psi = \ln(1 + 0.9082 \times 10^{-15} \frac{P^7}{m^8 R^3} \omega\theta^3) \tag{5.37}$$

$$\psi = \ln(1 + 0.118 \text{ R}\epsilon) \tag{5.38}$$

and from equation 2.38 the Rosseland radiative mean free path:

$$L_R = 0.60 \times 10^{-25} \frac{P^9}{m^{11} R^{7/2}} \frac{\omega^2 \theta^{7/2}}{\psi'} \tag{5.39}$$

where P, m and R are as defined in Chapter II, and

$$\psi' = 1 - 0.00203 \frac{P^{2/3}}{m^{2/3} R^{1/3}} \theta^{1/3} \tag{5.40}$$

Formula 5.39 holds only for $\psi' > 0$, or

$$T \lesssim 10^{8}\ {}^{o}K \tag{5.41}$$

The ratio of L_R to L_P

$$\frac{L_R}{L_P} \simeq 10^{-11} \frac{P^4}{m^5 R} \omega^2 \theta \frac{\psi}{\psi'} \tag{5.42}$$

is, for the region of interest, between 10^6 and 10^{10} (see Appendix D), and, from equations 5.21, $L_Q \gg L_R$, since $\beta(\sim \frac{v}{c})$ appears in the denominator in the second term. Thus:

$$\delta' = \frac{L_P}{L_Q} \ll 1 \tag{5.43}$$

The basic features of the shock structure problem will not be affected if it is assumed that L_P, L_Q and L_R are constant. Reasonable values for these constants are:

$$\bar{L}_P = (L_{Po} L_{P1})^{1/2} \tag{5.44}$$

$$\bar{L}_Q = (L_{Qo} L_{Q1})^{1/2} \tag{5.45}$$

$$\bar{L}_R = (L_{Ro} L_{R1})^{1/2} \tag{5.46}$$

where the subscript 0 applies to the pre-shock state and the subscript 1 applies to the post-shock state. If $\delta(\ll 1)$ is defined as

$$\delta = \frac{\bar{L}_P}{\bar{L}_Q}$$

Equation 5.22 can be written

$$\frac{d\theta}{d\omega} = \delta \frac{G}{F} \tag{5.47}$$

It can now be shown that the Eddington approximation is valid if, at all points along the shock curve $\omega = \omega(x)$, $\theta = \theta(x)$,

$$| F(\omega, \theta) | \leq \mathcal{O}(\delta) \tag{5.48}$$

and

$$| \bar{L}_Q \frac{dF}{dx}(\omega, \theta) | \leq \mathcal{O}(\delta) \tag{5.49}$$

First, since L_R is the mean of the $L_{R\nu}$, defined according to equation 5.15 in terms of a smooth function of ν, it can be assumed that if

$$| \bar{L}_R \frac{d^2}{dx^2} | \ll 1 \tag{5.50}$$

then $| L_{R\nu}{}^2 \frac{d^2}{dx^2} | \ll |$ for all ν, i.e., if 5.50 holds, the Eddington approximation is valid. The operator $\frac{d}{dx}$ can be written

$$\frac{d}{dx} = \left(\frac{d\omega}{dx}\frac{\partial}{\partial\omega} + \frac{d\theta}{dx}\frac{\partial}{\partial\theta}\right)$$

or, according to equations 5.19 and 5.20

$$\frac{d}{dx} = \left(\frac{F}{\overline{L}_P}\frac{\partial}{\partial\omega} + \frac{G}{\overline{L}_Q}\frac{\partial}{\partial\theta}\right)$$

then 5.50 can be written

$$\overline{L}_R{}^2 \left| \frac{G^2}{\overline{L}_Q{}^2}\frac{\partial^2}{\partial\theta^2} + \frac{2FG}{\overline{L}_P\overline{L}_Q}\frac{\partial^2}{\partial\omega\partial\theta} + \frac{F^2}{\overline{L}_P{}^2}\frac{\partial^2}{\partial\omega^2} + \left(\frac{GG_\theta}{\overline{L}_Q{}^2} + \frac{FG_\omega}{\overline{L}_P\overline{L}_Q}\right)\frac{\partial}{\partial\theta} + \frac{1}{\overline{L}_P}\frac{dF}{dx}\frac{\partial}{\partial\omega} \right| \ll 1 \tag{5.51}$$

where

$$G_\theta = \frac{\partial G}{\partial\theta}, \text{ etc.}$$

or

$$\frac{\overline{L}_R{}^2}{\overline{L}_Q{}^2} \left| G^2\frac{\partial^2}{\partial\theta^2} + \frac{2FG}{\delta}\frac{\partial^2}{\partial\omega\partial\theta} \frac{F^2}{\delta^2}\frac{\partial^2}{\partial\omega^2} + \left(GG_\theta + \frac{FG_\omega}{\delta}\right)\frac{\partial}{\partial\theta} + \frac{\overline{L}_Q}{\delta}\frac{dF}{dx}\frac{\partial}{\partial\omega} \right| \ll 1 \tag{5.52}$$

Now $\frac{\overline{L}_R{}^2}{\overline{L}_Q{}^2} = \mathcal{O}(\beta^2) \ll 1$, so that if the terms multiplying this quantity are of order 1, inequality 5.52 holds. In the region of interest, G is at most of order 1 along the shock curve (see Appendix D), and if the functions operated on (e.g., F, G) are themselves of at most order 1, and are sufficiently smooth, the partial derivatives are at most of order 1. Then if 5.48 and 5.49 hold, 5.50 must hold, and the Eddington approximation is valid.

An equivalent statement of equation 5.48 is

$$\bar{L}_R \ll t_\omega \tag{5.53}$$

where t_ω is the shock thickness with respect to the dimensionless velocity ω.

The shock thickness with respect to a particular variable y, t_y, is defined in the following manner: Let the variable y have pre-shock and post-shock values (obtained from the solution of the jump equations) y_o and y_1, respectively. If $y_o \neq y_1$,

$$t_y \equiv \frac{|y_o - y_1|}{\left|\frac{dy}{dx}\right|_{max}} \tag{5.54}$$

where $\left|\frac{dy}{dx}\right|_{max}$ is the maximum absolute value of $\frac{dy}{dx}$ within the shock, determined from the solution of the shock differential equations (see Fig. 5.1). If $y_o = y_1$ (as for instance, in this case $\phi_{1,o} = \phi_{1,1} = 0$), let Δy be the maximum absolute value of $y - y_o$ within the shock (see Fig. 5.2). Then

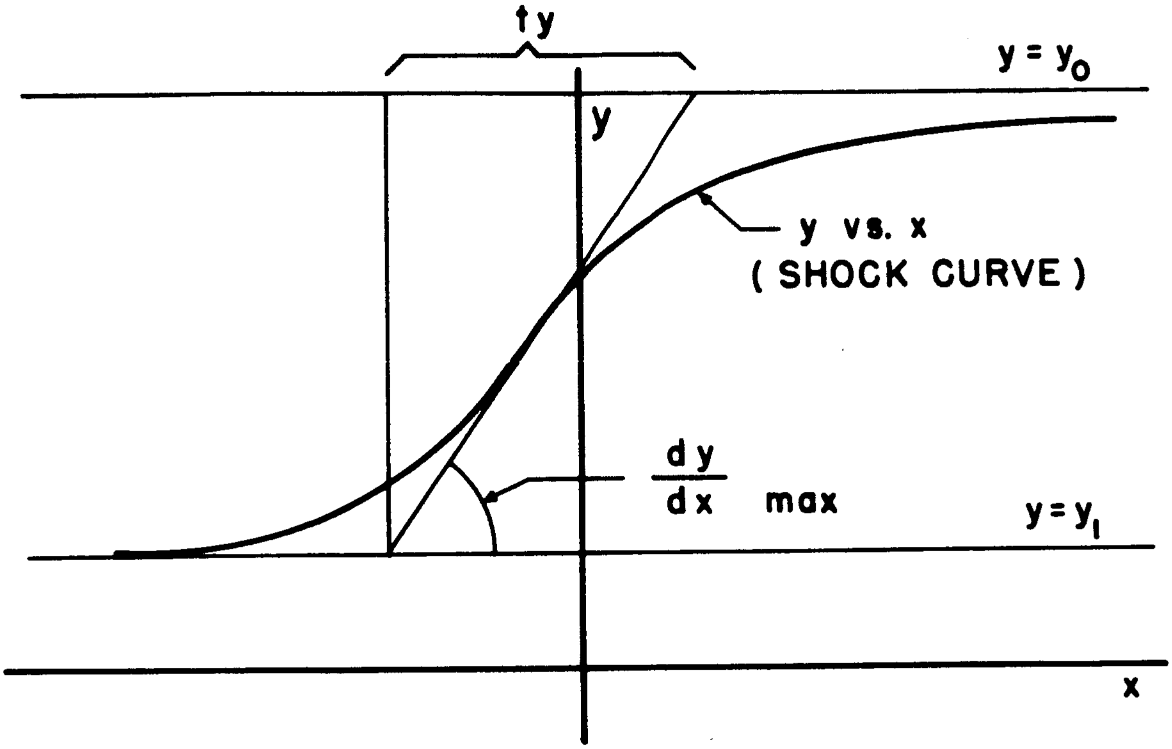


FIG. 5.1

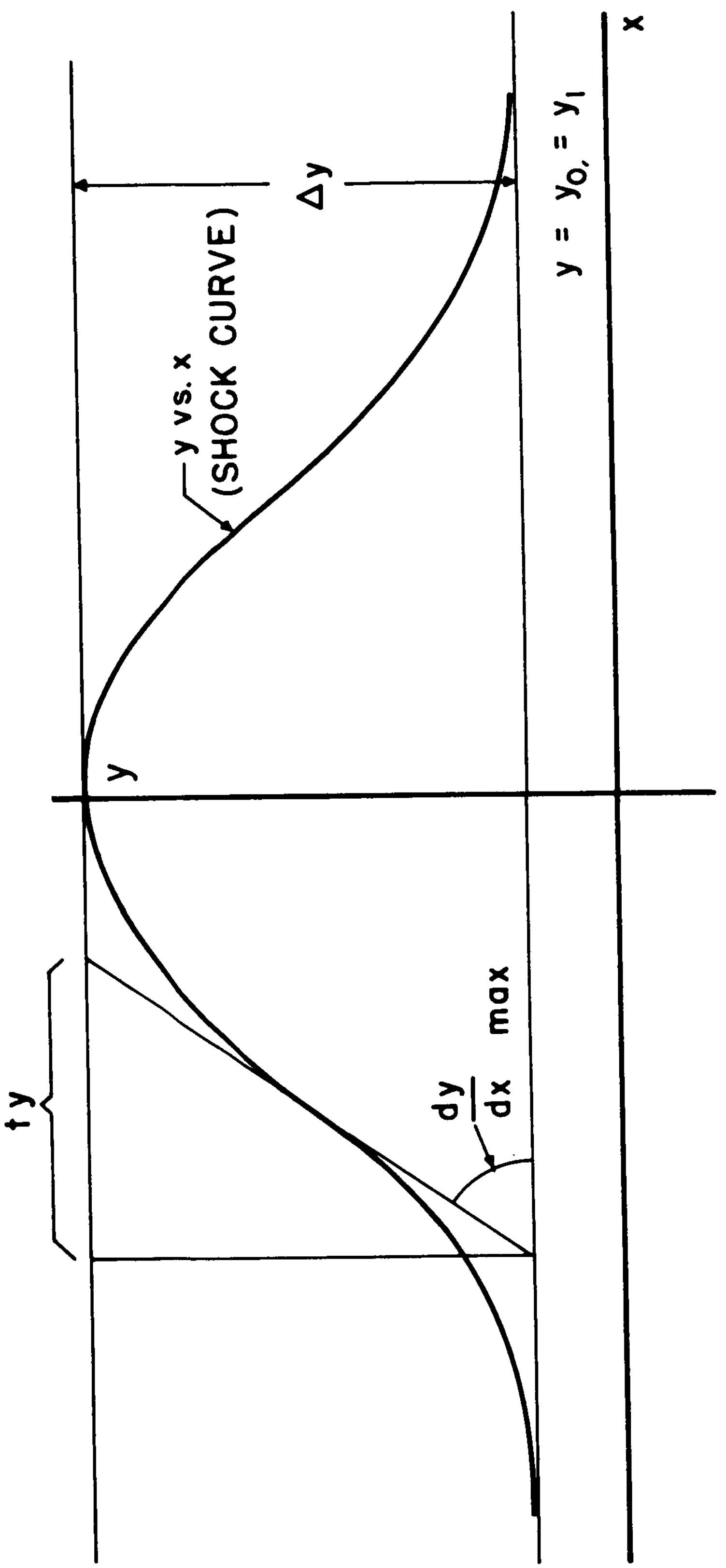


FIG. 5.2

$$t_y \equiv \frac{\Delta y}{\left|\frac{dy}{dx}\right|_{max}} \tag{5.55}$$

By equations 5.19 and 5.54

$$t_\omega = \frac{\bar{L}_P |\omega_o - \omega_1|}{|F|_{max}}$$

where $|F|_{max}$ is the maximum absolute value of F along the shock curve. From 5.48

$$|F|_{max} = \mathcal{O}\left(\frac{\bar{L}_P}{\bar{L}_Q}\right)$$

$$\frac{\bar{L}_R}{t_\omega} = \left(\frac{\bar{L}_R}{\bar{L}_Q}\right) \frac{1}{|\omega_o - \omega_1|} = \mathcal{O}(\beta) \ll 1 \tag{5.56}$$

since $|\omega_o - \omega_1|$ is of order 1.

It will also be true that

$$\bar{L}_R \ll t_\theta \tag{5.57}$$

where t_θ is the shock thickness with respect to the dimensionless temperature θ, since by equations 5.20 and 5.54

$$t_\theta = \frac{\bar{L}_Q |\theta_o - \theta_1|}{|G|_{max}}$$

and thus

$$\frac{\bar{L}_R}{t_\theta} = \left(\frac{\bar{L}_R}{\bar{L}_Q}\right) \frac{|G|_{max}}{|\theta_o - \theta_1|} = \mathcal{O}(\beta) \ll 1 \tag{5.58}$$

since $|G|_{max}$ and $|\theta_o - \theta_1|$ can both be considered of order 1. Thus, when the Eddington approximation holds, the radiative mean free path is smaller than the characteristic lengths in which the gas dynamic variables change through the shock, and radiation emitted in the shock will be reabsorbed within the shock.

Equations 5.19 and 5.20 are of the form:

$$L(y) \frac{dy}{dx} = F(y) \tag{5.59}$$

where $L(y)$ is a diagonal matrix, in this case

$$\begin{pmatrix} L_P & 0 \\ 0 & L_Q \end{pmatrix}$$

y is a column vector, in this case

$$\begin{pmatrix} \omega \\ \theta \end{pmatrix}$$

and $F(y)$ is a line vector, in this case $(F(\omega, \theta), G(\omega, \theta))$.

The boundary conditions on the problem are that y is known at two points, P_o and P_1, such that:

$$F(y_o) = F(y_1) = 0 \tag{5.60}$$

Since the right-hand side of 5.59 does not contain x explicitly, and condition 5.60 applies at points P_o and P_1 which are the only known points on the shock curve, equation 5.59 cannot be integrated directly. That is, a Taylor series:

$$y(x_o + \Delta x) = y_o + \frac{dy}{dx}\bigg|_{y_o} \Delta x + \frac{d^2y}{dx^2}\bigg|_{y_o} \frac{\Delta x^2}{2!} + \ldots \tag{5.61}$$

will have all zeros on the right-hand side except for the first term (if Δx is finite) and

$$y(x_o + \Delta x) = y_o \tag{5.62}$$

for all finite Δx. Points P_o and P_1 are called equilibrium points of 5.59, and because of equation 5.60 x_o and x_1, must have infinite values (either $+\infty$ or $-\infty$).

The problem of numerical integration of equation 5.59 is discussed in Appendix C. At present it suffices to mention that equation 5.59 is linearized in the neighborhood of P_o or P_1:

$$L \frac{dy}{dx} = M(y-y_o) \tag{5.63}$$

where M is a matrix whose components are

$$M_{ij} = \frac{\partial F_i}{\partial y_j} \tag{5.64}$$

and a change of coordinates performed to reduce 5.63 to

$$L\,\frac{dy}{dx} = My \tag{5.65}$$

If, in the present case, $\lambda^{o}_{1,2}$ and $\lambda^{1}_{1,2}$ are the solutions of the equation

$$\det\,|\,M - \lambda L\,| = 0 \tag{5.66}$$

at P_o and P_1 respectively, it can be shown that (see Appendix C): λ^o_1 and λ^o_2 are both real and positive; λ^1_1 and λ^1_2 are real and of opposite signs, say $\lambda^1_1 < 0$, $\lambda^1_2 > 0$; there is a unique solution curve to equations 5.19 and 5.20, for which P_o (the pre-shock state) corresponds to $x = -\infty$ and P_1 (the post-shock state) corresponds to $x = +\infty$; and the slope of that curve in the phase plane at P_1 is:

$$\frac{\Delta\theta}{\Delta\omega} = -\,\frac{M_{21}}{M_{22} - \lambda^1_1 L_Q} = -\,\frac{M_{11} - \lambda^1_1 L_P}{M_{12}} \tag{5.67}$$

where, according to equation 5.64,

$$M_{11} = F_\omega = 1 - \frac{\theta}{\omega^2} \tag{5.68}$$

$$M_{12} = F_\theta = \frac{1}{\omega} + \frac{4}{3}\,S\theta^3 \tag{5.69}$$

$$M_{21} = G_\omega = 2D(S\theta^4 + (1 - \omega)) \tag{5.70}$$

$$M_{22} = G_\theta = 1 + 8DS\omega\theta^3 \tag{5.71}$$

the matrix elements being evaluated at P_1. It is also shown in Appendix C that the solution curve with slope 5.67 will lie in that region of the phase plane defined by:

$$F(\omega, \theta) < 0 \; ; \; G(\omega, \theta) > 0 \tag{5.72}$$

(See Figs. 5.3 and 5.4)

Since the values of ω and θ at a point on the solution curve near P_1 $(\omega = \omega^1, \theta = \theta^1)$ can be determined from equation 5.67, equations 5.19 and 5.20 can now be numerically integrated on the phase plane, using the quotient equation 5.47, $\frac{d\theta}{d\omega} = \delta \frac{G}{F}$.

However, advantage can be taken of the fact that $\delta \ll 1$ to eliminate the need for numerical integration. Since $\delta \ll 1$, equation 5.47 immediately reveals two interrelated facts. Assuming $\frac{G}{F}$ is of order 1 at some point on the shock curve, then as the integration progresses in the neighborhood of that point, θ will change much less than ω, i.e.,

$$\left| \frac{\Delta\theta}{\Delta\omega} \right| \ll 1 \tag{5.73}$$

Figure 5.5 shows that if this situation continues for some distance, the integral curve will approach the curve $F(\omega,\theta)=0$. Alternatively, since the change in θ along the integral curve from P_1 to P_o is finite, the curve must be such that for at least part of it

$$\left| \frac{G}{F} \right| \ll 1 \tag{5.74}$$

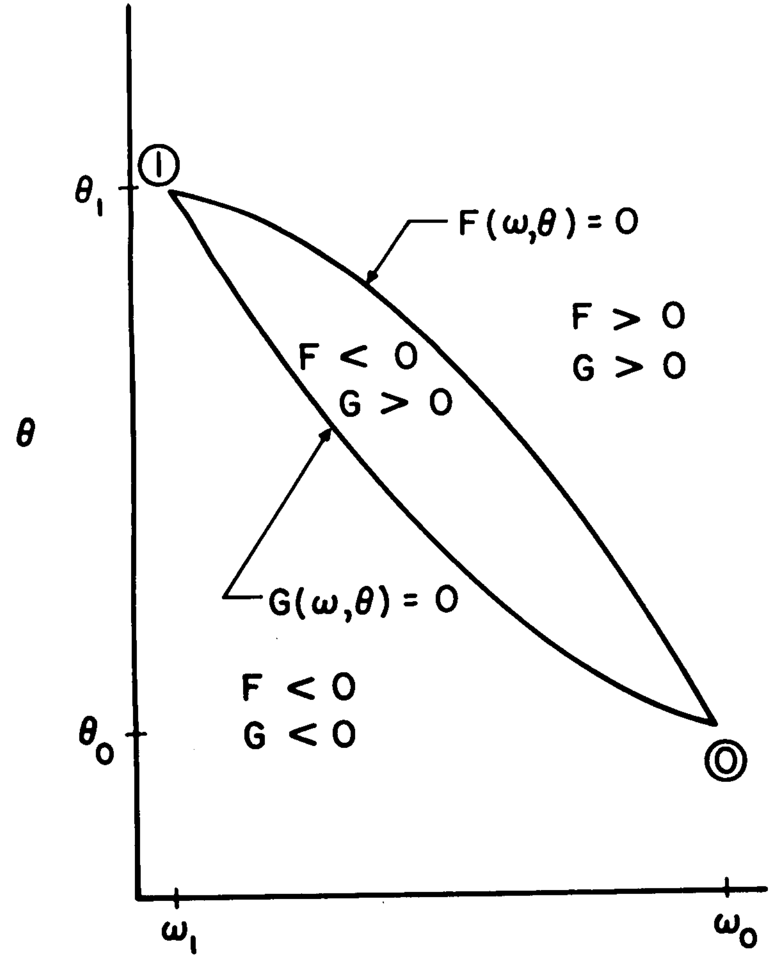

θ
θ₁
θ₀
ω₁
ω₀
1
0
F(ω,θ) = 0
G(ω,θ) = 0
F > 0
G > 0
F < 0
G > 0
F < 0
G < 0

FIG. 5.3

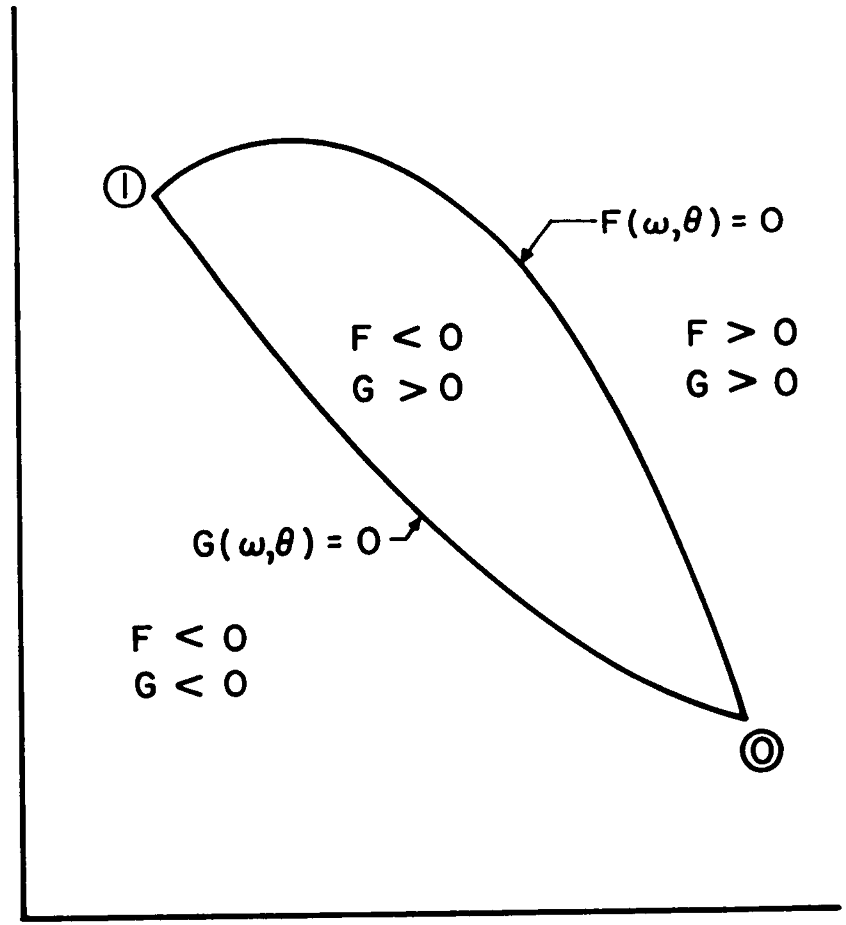

1
0
F(ω,θ) = 0
G(ω,θ) = 0
F < 0
G > 0
F > 0
G > 0
F < 0
G < 0

FIG. 5.4

An integral curve of 5.47 must thus satisfy 5.74 for some range of values of the parameter x, and if values of x exist for which 5.74 is not satisfied, 5.73 must be satisfied for these values.

Two cases must now be distinguished, according to the sign of F_ω at point P_1:

$$\text{Case A:} \quad F_\omega(P_1) > 0 \tag{5.75}$$

$$\text{Case B:} \quad F_\omega(P_1) < 0 \tag{5.76}$$

The slope in the phase plane of the curve $F = 0$ at point P_1 is

$$\left.\frac{d\theta}{d\omega}\right|_{F=0} = -\frac{F_\omega(P_1)}{F_\theta(P_1)} \tag{5.77}$$

and thus is negative in Case A and positive in Case B, since F_θ is always positive according to equation 5.69. G_ω and G_θ are also always positive. Since (see Chapter III)

$$\frac{1}{M^2} = \frac{\gamma'\theta}{\omega^2} \tag{5.78}$$

F_ω can be written

$$F_\omega = 1 - \frac{1}{\gamma' M^2} \tag{5.79}$$

Since P_o represents the pre-shock state, for which $M > 1$, F_ω is always positive at that point. Thus, two possible shapes exist for the region defined by the inequalities 5.72, as shown in Figs. 5.3, corresponding to Case A, and 5.4 corresponding to Case B. The value of λ_1^1 from equation 5.66 and the slope of the integral curve at P_1 from equation 5.67 can be computed for each of the two cases with the following results to order δ:

$$\text{Case A:} \quad \frac{\Delta\theta}{\Delta\omega} = - \frac{F_\omega}{F_\theta} + \mathcal{O}(\delta^2) \tag{5.80}$$

$$\text{Case B:} \quad \frac{\Delta\theta}{\Delta\omega} = \delta \frac{G_\omega}{F_\omega} + \mathcal{O}(\delta^2) \tag{5.81}$$

Thus, to order δ, the integral curve for Case A follows the curve $F = 0$ at point P_1, and for Case B the slope of the integral curve at point P_1 is negative but almost horizontal, since $\delta \ll 1$. In other words, in Case A, the integral curve satisfies 5.74 at point P_1, while in Case B it satisfies 5.73.

Now 5.74 is a stable condition of equation 5.47 in the direction from P_1 to P_o (i.e., of increasing ω), provided F_ω is positive. That is (see Fig. 5.5), if F tends to increase in absolute value as the integration proceeds in that direction, $\left| \frac{d\theta}{d\omega} \right|$ tends to decrease, so that the integral curve is forced into a direction for which $| F |$ tends to

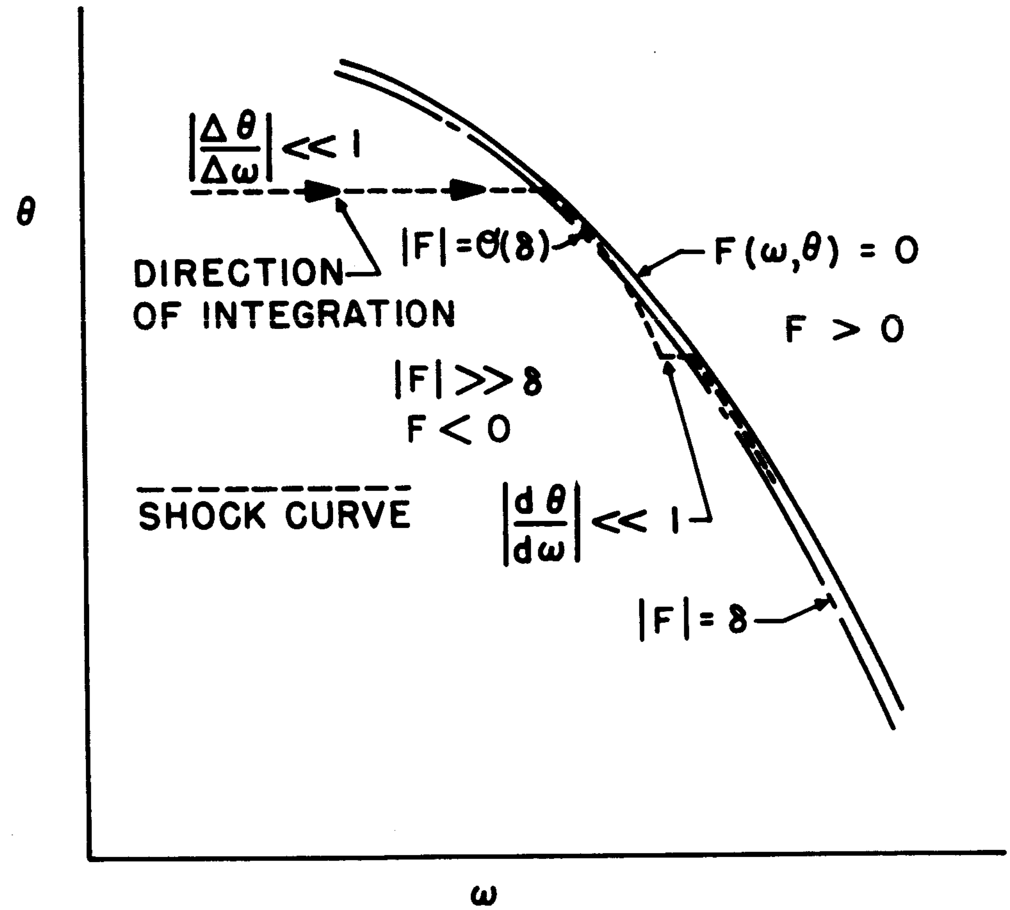


FIG. 5.5 TENDENCY OF SHOCK CURVE TO FOLLOW $F(\omega,\theta) = 0$

decrease. Thus, once the curve satisfies 5.74 for some value of x, x^*, it will satisfy 5.74 for all $x > x^*$. In Case A this means that 5.74 will be satisfied for all x (see Fig. 5.6). A more exact statement of 5.74 can be found. Since equation 5.47 is true, and near the curve $F = 0$, $\left| \frac{d\theta}{d\omega} \right|$ along the shock curve is of order 1, while G is at most of order 1 (see Appendix D), 5.74 can be expressed as

$$| F | \leq \mathcal{O}(\delta) \tag{5.82}$$

This is identical to 5.48, showing that for Case A the shock curve staisfies 5.48 at all points.

In Case B the integral curve must proceed from P_1 with decreasing x in an almost horizontal direction (see Fig. 5.7) until it reaches a region where 5.82 is satisfied, whence it must closely parallel the curve $F = 0$ until point P_o is reached.

In both cases the shock curve in the phase plane can thus be found to close approximation without numerically integrating. In Case A the curve follows the curve $F = 0$ from P_1 to P_o (Fig. 5.6). In Case B the curve lies along the line $\theta = \theta^1$ from P_1 until that line once again intersects the curve $F = 0$, and then follows that curve to P_o (Fig. 5.7).

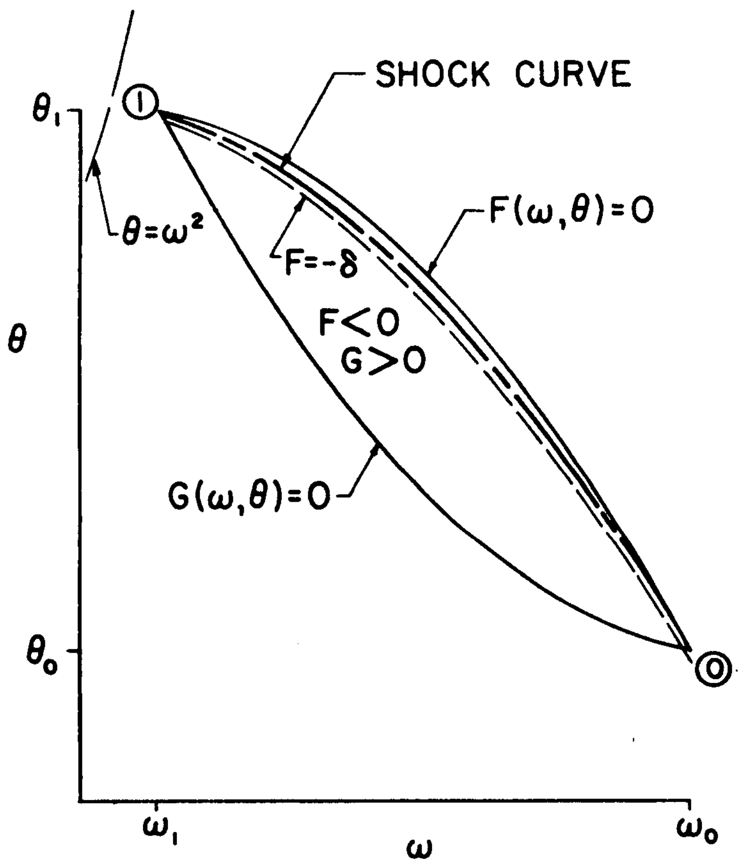


FIG. 5.6 CASE A
$\theta_1 < \omega_1^2 \quad (M_1^2 > \frac{1}{\gamma_1'})$

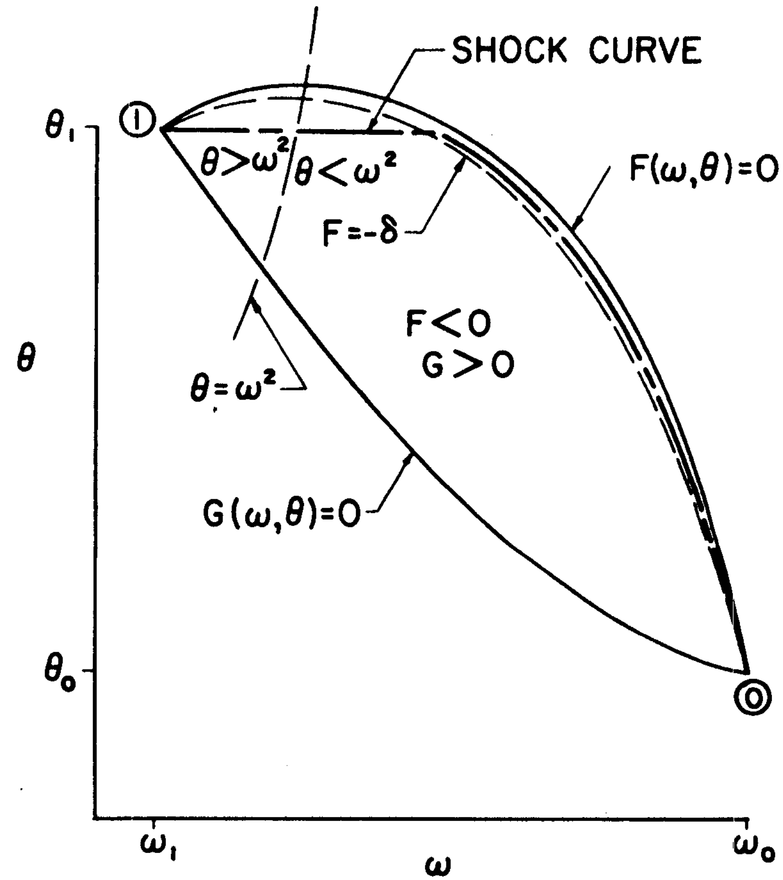


FIG. 5.7 CASE B
$\theta_1 > \omega_1^2 \quad (M_1^2 < \frac{1}{\gamma_1'})$

PHASE PLANE PLOTS OF SHOCK CURVE $\frac{d\theta}{d\omega} = \delta \frac{G(\omega,\theta)}{F(\omega,\theta)}$

A-907-ME-0170

It can now be shown that in Case A the Eddington approximation holds. Inequality 5.48 has been shown to apply. From equation 5.19 $L_P \frac{d\omega}{dx} = \mathcal{O}(\delta)$, so that

$$L_Q \frac{d\omega}{dx} = \mathcal{O}(1) \tag{5.83}$$

also

$$L_Q \frac{d\theta}{dx} = \mathcal{O}(1) \tag{5.84}$$

so that the characteristic length of all the variables in Case A is L_Q. If 5.49 is not satisfied, i.e., if

$$\left| \bar{L}_Q \frac{dF}{dx} \right| \gg \delta \tag{5.85}$$

then in one characteristic length

$$\left| \Delta F \right| \gg \delta \tag{5.86}$$

so that 5.48 cannot be satisfied. Therefore, since 5.48 and 5.49 are both valid for Case A, the Eddington approximation holds in that case. A sufficient condition for the validity of the Eddington approximation is thus:

$$F_\omega(P_1) > 0$$

This criterion can be expressed in a number of ways, for instance,

$$\theta^1 < \omega^{1^2} \tag{5.87}$$

or

$$\frac{1}{M_1^2} < \gamma_1' \tag{5.88}$$

Inequality 5.88 is especially useful in the physical interpretation of this result, since it shows two differenct circumstances in which the Eddington approximation is valid, namely:

(1) If the shock is very weak, so that M_1^2 is comparatively large (e.g., for $\epsilon_1 \sim 0$, if $M_1 > \sqrt{0.6}$).

(2) If the radiation pressure in the post shock state is so dominant that γ_1' is large (e.g., for $\epsilon_1 = 18$, $\gamma_1' \sim 8$, so that $\gamma_1' > \frac{1}{M_1^2}$ for any possible M_1).

By means of the jump equation, 5.87 or 5.88 can be transferred to pre-shock conditions, resulting in Fig. 5.8, which shows the regions in the $M_o - \epsilon_o$ plane in which the Eddington approximation is valid. It can be seen from this figure that for $\epsilon_o \gtrsim 4.2$, or for $M_o \lesssim 1.3$, the approximation always holds, and that as the Mach number increases above ~ 3.5 or decreases below that value, the approximation holds for smaller values of ϵ_o. There is thus only a limited region of low ϵ_o and intermediate M_o for which the Eddington approximation is not valid.

The shock curves in the phase plane can be transferred to real space (i.e., T and v vs. x), as is done in Fig.

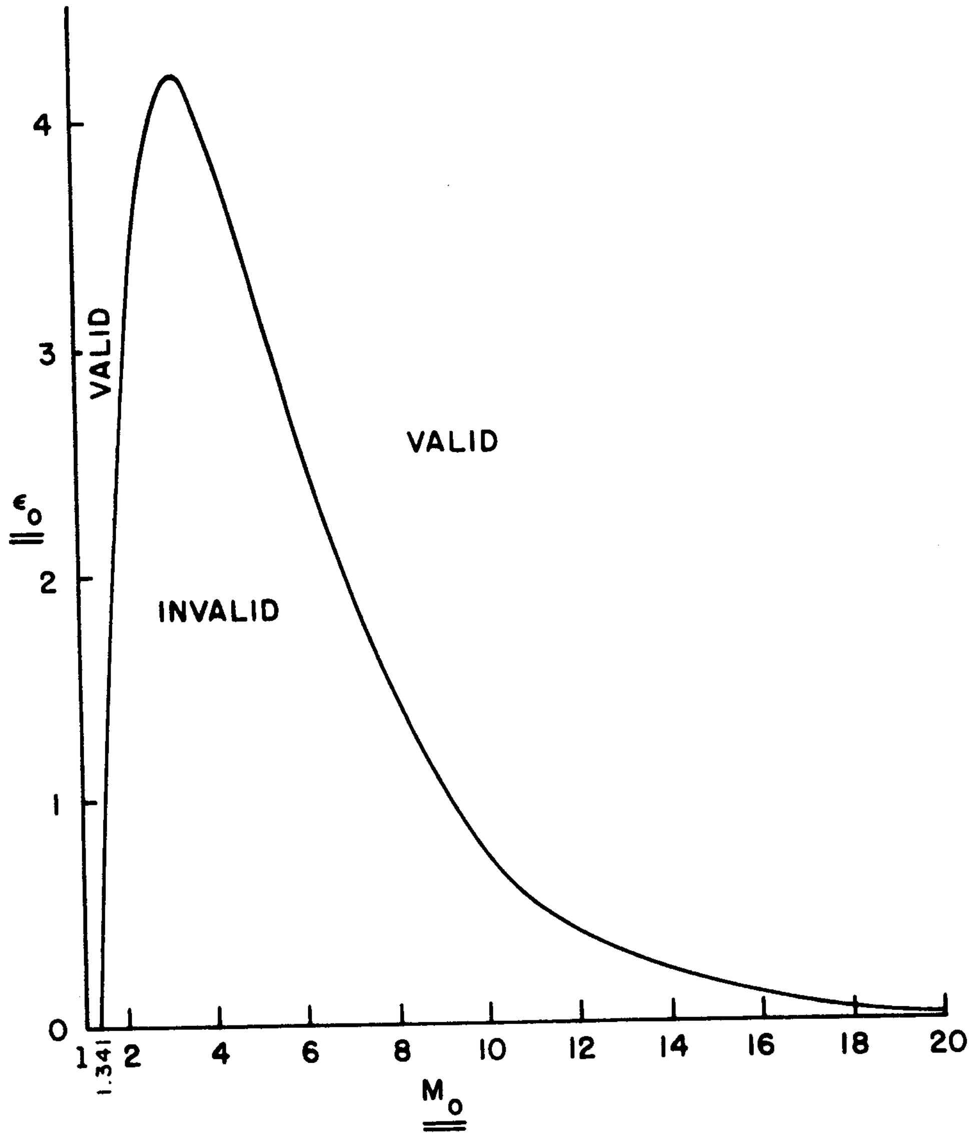


FIG. 5.8 VALIDITY OF DIFFUSION APPROX IN $M_0 - \epsilon_0$ PLANE.

5.9. From Fig. 5.9 it is clear that when the Eddington approximation is valid both T and v vary over a charachteristic length L_Q, and that for Case B equations 5.19 and 5.20 lead to characteristic lengths L_P for velocity (and therefore density) and L_Q for temperature. However, the results shown for Case B are mathematical only, since one of the physical assumptions used to obtain them is invalid. Their correspondence, if any, with physical reality can only be determined by solution of the equations of the first moment approximation.

It is shown in Appendix C that a solution curve to these equations exists. However, the determination of this curve presents great difficulty.

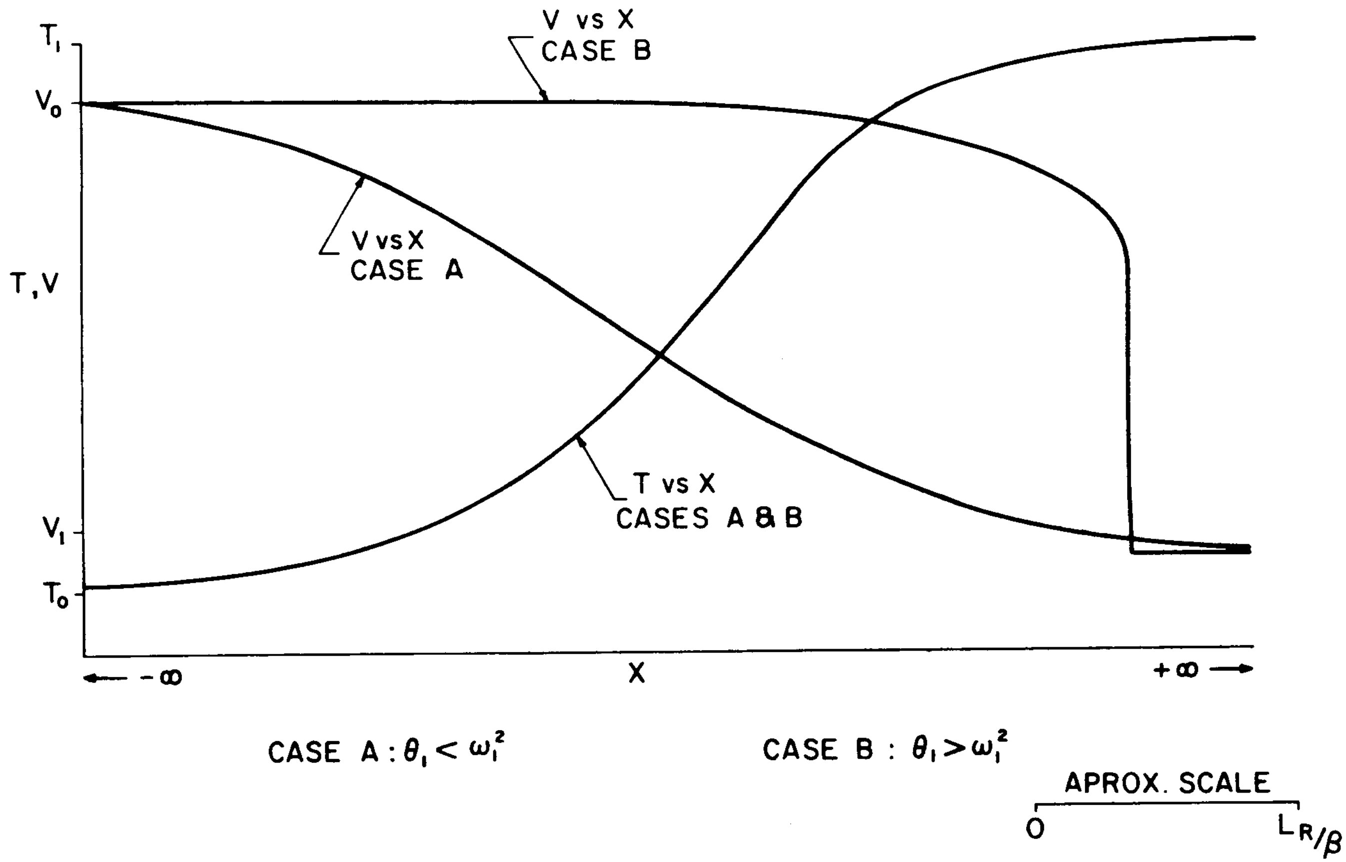


FIG. 5.9 REAL SPACE SHOCK PROFILES (T & V vs X)

APPENDIX A

THE SOLUTIONS OF THE JUMP EQUATION

In this appendix equation 3.27:

$$\frac{\omega^2 G(\omega)}{(F(\omega))^4} = \frac{1}{K} \tag{A1}$$

is studied and the following results are obtained:

(1) There are only two physically possible roots, ω^o and ω^1 ($\omega^o \geq \omega^1$) which correspond to the pre- and post-shock states respectively.

(2) ω^o lies between the larger of the two positive roots of $G(\omega) = 0$ and the larger of the two positive roots of $F(\omega) = 0$.

(3) ω^1 lies between the smaller of the two positive roots of $G(\omega) = 0$ and the smaller of the two positive roots of $F(\omega) = 0$.

According to the definitions in Chapter III

$$F(\omega) = \omega(1 - \omega)(1 - 7\omega) + (A\omega + 2Z) \tag{A2}$$

$$G(\omega) = \omega(1 - \omega)(1 - g\omega) + (A\omega + hZ) \tag{A3}$$

where ω is the dimensionless velocity, $\omega = \frac{mv}{P}$, and g and h are functions of the specific heat ratio γ

$$g(\gamma) = \frac{\gamma + 1}{\gamma - 1} \tag{A4}$$

$$h(\gamma) = \frac{2 - \gamma}{\gamma - 1} \tag{A5}$$

The constant K in equation A1 is

$$K = \frac{3f^3}{S} \tag{A6}$$

where $$f(\gamma) = 2\left(\frac{3\gamma - 4}{\gamma - 1}\right) \tag{A7}$$

The roots of A1 thus depend on γ and on the three shock parameters:

$$A = \frac{2Qm}{P^2} - 1 \tag{A8}$$

$$S = \frac{aP^7}{m^8R^4} \tag{A9}$$

$$Z = \frac{E^2m^2}{\mu P^3} \tag{A10}$$

The constants on the right-hand side of these equations are defined in Chapter II.

The dimensionless temperature $\theta = \frac{m^2RT}{P^2}$ can be found from equation A1 using equation 3.11 or 3.12

$$(f\theta)^4 = \frac{KG(\omega)}{\omega^2} \tag{A11}$$

$$f\theta = -\frac{F(\omega)}{\omega} \tag{A12}$$

It is to be noted that for $\gamma = \frac{4}{3}$ $F(\omega) = G(\omega)$ and $f(\gamma) = 0$, so that A1 reduces to

$$F(\omega) = 0 \tag{A13}$$

which is a mathematically degenerate case. However, even in this case there are two physical roots, namely, the two positive roots of A13.

For the purposes of this appendix equation A1 will be studied for $\gamma = \frac{5}{3}$, so that

$$f(\gamma) = 3 \tag{A14}$$

$$g(\gamma) = 4 \tag{A15}$$

$$h(\gamma) = \frac{1}{2} \tag{A16}$$

$$G(\omega) = \omega(1 - \omega)(1 - 4\omega) + (A\omega + \frac{1}{2} Z) \tag{A17}$$

This is the case of real physical interest, but mathematically the results obtained for $\gamma = \frac{5}{3}$ apply, with numerical differences, to all $\gamma > \frac{4}{3}$. An analysis can be made for $\gamma < \frac{4}{3}$, the results of which are quite similar to those obtained here, but unimportant for the problem at hand.

The special case of $Z = 0$ (no magnetic field) will be considered first, and then the general case $Z \neq 0$ will be studied.

I Case Z = 0

Equation A1 reduces to

$$\frac{G_1(\omega)}{\omega(F_1(\omega))^4} = \frac{1}{K} \tag{A18}$$

where G_1 and F_1 are quadratic expressions:

$$G_1(\omega) = (1 - \omega)(1 - 4\omega) + A \tag{A19}$$

$$F_1(\omega) = (1 - \omega)(1 - 7\omega) + A \tag{A20}$$

Equations A11 and A12 become (for $f = 3$)

$$(3\theta)^4 = \frac{KG_1(\omega)}{\omega} \tag{A21}$$

$$3\theta = - F_1(\omega) \tag{A22}$$

so that, since $\theta \geq 0$ for physical solutions, these solutions must have

$$G_1(\omega) \geq 0 \tag{A23}$$

$$F_1(\omega) \leq 0 \tag{A24}$$

Now if G_1 has real roots, say $\bar{\omega}_g^o$ and $\bar{\omega}_g^1 (\bar{\omega}_g^o \geq \bar{\omega}_g^1)$, then A23 is satisfied on the real ω axis for

$$\bar{\omega}_g^1 \geq \omega \tag{A25}$$

and $$\omega \geq \overline{\omega}_g^o \tag{A26}$$

while if the roots of G_1 are complex A23 is satisfied for all real ω. If F_1 has real roots, say $\overline{\omega}_f^o$ and $\overline{\omega}_f^1 (\overline{\omega}_f^o \geq \overline{\omega}_f^1)$, then A24 is satisfied on the real ω axis for

$$\overline{\omega}_f^o \geq \omega \geq \overline{\omega}_f^1 \tag{A27}$$

while if the roots of F_1 are complex there is no real ω which satisfies A24. Then for any physical problem the roots of F_1 must be real. Since

$$\overline{\omega}_f^{o,1} = \frac{4 \pm \sqrt{9 - 7A}}{7} \tag{A28}$$

there is a maximum value of A in physical cases,

$$A \leq \frac{9}{7} \tag{A29}$$

Also, combining A25, A26 and A27, the physical solutions of A18 must satisfy

$$\overline{\omega}_f^o \geq \omega \geq \overline{\omega}_g^o \tag{A30}$$

$$\overline{\omega}_g^1 \geq \omega \geq \overline{\omega}_f^1 \tag{A31}$$

if the roots of G_1 are real. A30 and A31 can only be satisfied in that case if

$$\bar{\omega}_f^o \geq \bar{\omega}_g^o \tag{A32}$$

$$\bar{\omega}_g^1 \geq \bar{\omega}_f^1 \tag{A33}$$

i.e., if the roots of G_1 lie inside the roots of F_1. To show that this is the case, equation A20 can be subtracted from A19 to give

$$G_1 - F_1 = 3\omega(1 - \omega) \tag{A34}$$

so that

$$G_1 - F_1 \geq 0 \quad \text{for} \quad 0 \leq \omega \leq 1$$

Then

$$G_1(\bar{\omega}_f^o) \geq 0$$

and

$$G_1(\bar{\omega}_f^1) \geq 0$$

since at those two points $F_1 = 0$. But this can only be true if

$$\bar{\omega}_f^o \geq \bar{\omega}_g^o$$

and

$$\bar{\omega}_g^1 \geq \bar{\omega}_f^1 \tag{A35}$$

i.e., if the roots of G_1 lie inside the roots of F_1.

Thus, if the roots of G_1 are complex, there is one physical region, R, defined by A27, while if the roots of G_1 are real, there are two physical regions, R_o and R_1, defined by A30 and A31 respectively.

It is now possible to show that the roots of A18 satisfy the following conditions for every A and S:

When the roots of G_1 are complex, there are two and only two real roots in R. When the roots of G_1 are real, there is one and only one real root in R_o (pre-shock state), and one and only one real root in R_1 (post-shock state). This is the equivalent of the statement: For every pre-shock state there exists one and only one post-shock state.

The proof of the above is based on the "root locus method" (Ref. 46):

Equation A18 can be rewritten:

$$\frac{(\omega - \overline{\omega}_g^o)(\omega - \overline{\omega}_g^1)}{\omega(\omega - \overline{\omega}_f^o)^4(\omega - \overline{\omega}_f^1)^4} = \frac{1}{K} = \frac{S}{81} \tag{A36}$$

Then, in the complex ω plane

$$\mathrm{Arg}(\omega-\overline{\omega}_g^o) + \mathrm{Arg}(\omega-\overline{\omega}_g^1) - \mathrm{Arg}(\omega) - 4\mathrm{Arg}(\omega-\overline{\omega}_f^o) - 4\mathrm{Arg}(\omega-\overline{\omega}_f^1)$$

$$= \mathrm{Arg}\,\frac{S}{81} = 0 \tag{A37}$$

In this expression $\overline{\omega}_g^o$ and $\overline{\omega}_g^1$ are the roots of the left-hand side and 0, $\overline{\omega}_f^o$ and $\overline{\omega}_f^1$ are its poles. For real ω, the argument of $(\omega-\omega_i)$, ω_i being a real root or pole, is π if $\omega < \omega_i$ and 0 if $\omega > \omega_i$. Thus a segment of the real ω axis to the left of an even number of real singular points (roots or poles) must be on a locus of roots of equation A36, the parameter of that locus being K.

For $K = 0$, the roots of A36 are 0, $\overline{\omega}_f^o$ and $\overline{\omega}_f^1$, the last two being roots of multiplicity 4. As $K \to \infty$, the roots of A36 approach $\overline{\omega}_g^o$ and $\overline{\omega}_g^1$. Since there are 9 poles and only two roots, seven of the root loci starting at the poles must terminate at the point at infinity. The set of loci must be symmetric with respect to the real axis, since the complex conjugate of any complex root must also be a root for the same value of K. The root loci which approach infinity to asymptotically to 7 lines at an angle of $\frac{2\pi}{7}$ to each other, all originating from the "center of gravity"

$$\omega_c = \frac{\overline{\omega}_g^o + \overline{\omega}_g^1 - 4(\overline{\omega}_f^o + \overline{\omega}_f^1)}{-7} \qquad \text{(A38)}$$

according to the general formula

$$\omega_c = \frac{\sum_i \omega_{oi} - \sum_i \omega_{pi}}{N_o - N_p} \qquad \text{(A39)}$$

where

ω_{oi} are the positions of the zeros

ω_{pi} are the positions of the poles

N_o and N_p are the number of zeros and poles, respectively.

As applied to A36 the method can be summarized graphically by two diagrams:

Case A: Roots of G_1 complex (Fig. A1)

Case B: Roots of G_1 real (Fig. A2)

(In these figures the x's represent poles and the O's represent zeros.)

From these diagrams, it can be seen that:

Case A

If for a given K there is one real root (corresponding to the pre-shock state) in R, then there is one and only one more real root (corresponding to the post-shock state) in R.

Case B

There is one and only one real root in R_o and one and only one real root in R_1 for any given K.

Thus there is one and only one post-shock state for every pre-shock state.

II Case $Z \neq 0$

The reasoning used here is similar to that in the previous case. In equation A1, $F(\omega)$ and $G(\omega)$ have three roots, and can be written in the general form

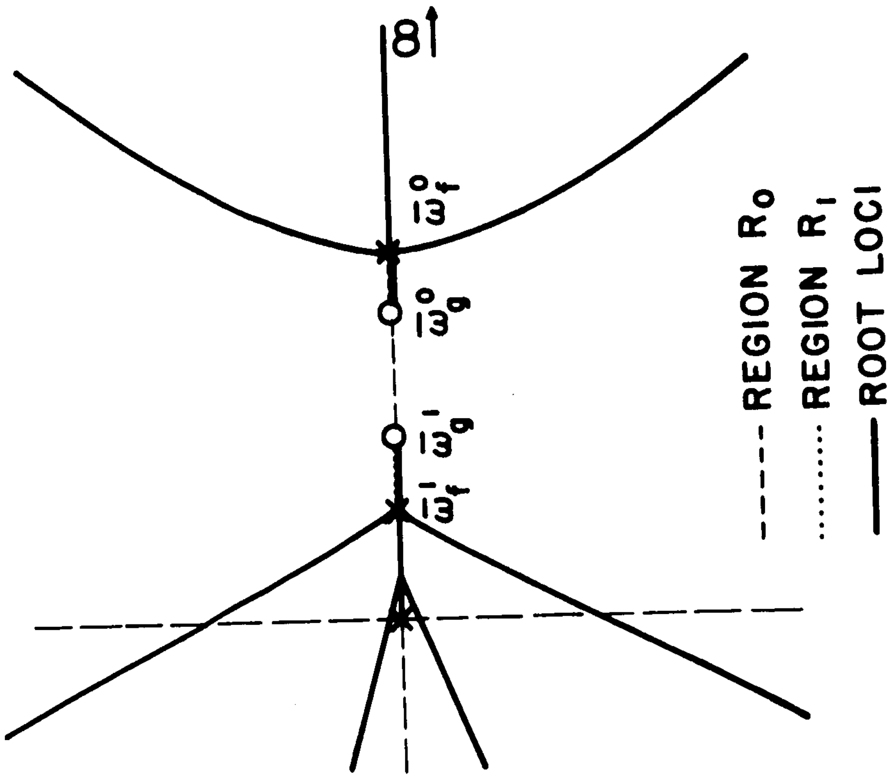

∞
$\bar{\omega}_f^0$
$\bar{\omega}_g^0$
$\bar{\omega}_g^1$
$\bar{\omega}_f^1$
REGION R_0
REGION R_1
ROOT LOCI

FIG. A2

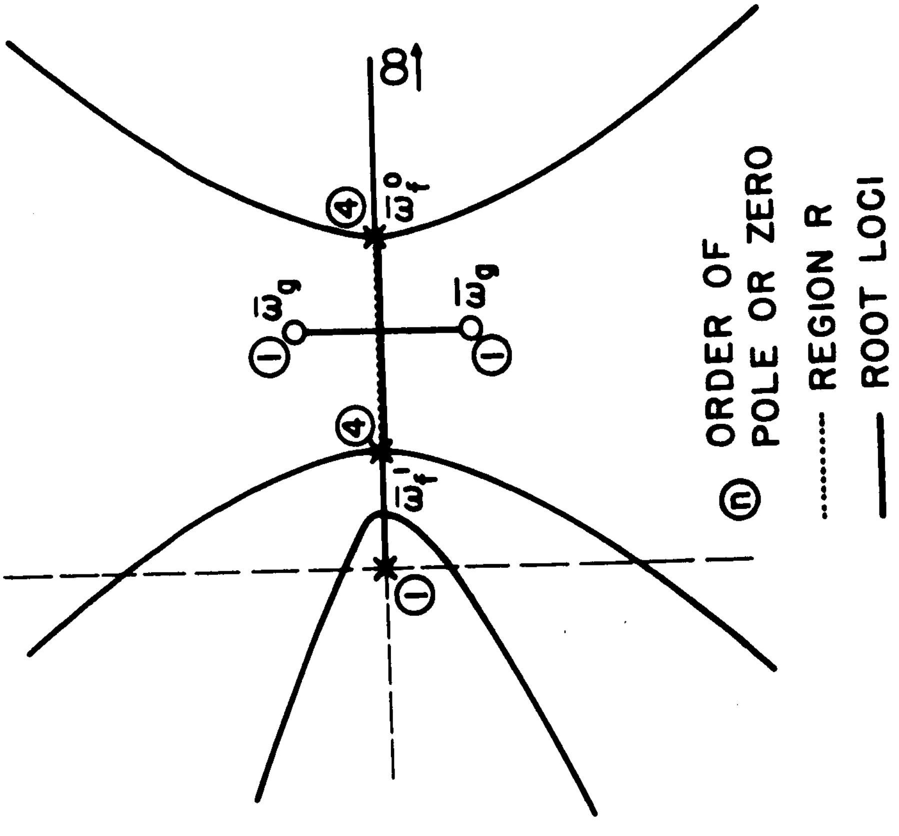

∞
4
$\bar{\omega}_f^0$
1
$\bar{\omega}_g$
$\bar{\omega}_g$
1
4
$\bar{\omega}_f^1$
1
n ORDER OF POLE OR ZERO
REGION R
ROOT LOCI

FIG. A1

$$F_\gamma(\omega) = M_g(\omega) + L_h(\omega) \tag{A40}$$

where
$$M_g(\omega) = \omega(1-\omega)(1-g\omega) \tag{A41}$$

and
$$L_h(\omega) = A\omega + hZ \tag{A42}$$

so

$$F(\omega) = F_{4/3}(\omega) \tag{A43}$$

$$G(\omega) = F_{5/3}(\omega) \tag{A44}$$

The roots of $F_\gamma(\omega)$ are the intersections of a cubic curve depending only on g as a parameter, and a line with slope $-A$ and intercept $-hZ$ (see Fig. A3), and can be labeled ω_f^-, ω_f^o, and ω_f^1 or ω_g^-, ω_g^o, and ω_g^1 for F and G respectively, or generically ω_γ^-, ω_γ^o, and ω_γ^1 as in Fig. A3.

For physical solutions, since $\theta \geq 0$

$$G(\omega) \geq 0 \tag{A45}$$

$$F(\omega) \leq 0 \tag{A46}$$

As before, if G has real positive roots, then A45 is satisfied on the real positive ω axis for

$$\omega_g^1 \geq \omega \tag{A47}$$

and
$$\omega \geq \omega_g^o \tag{A48}$$

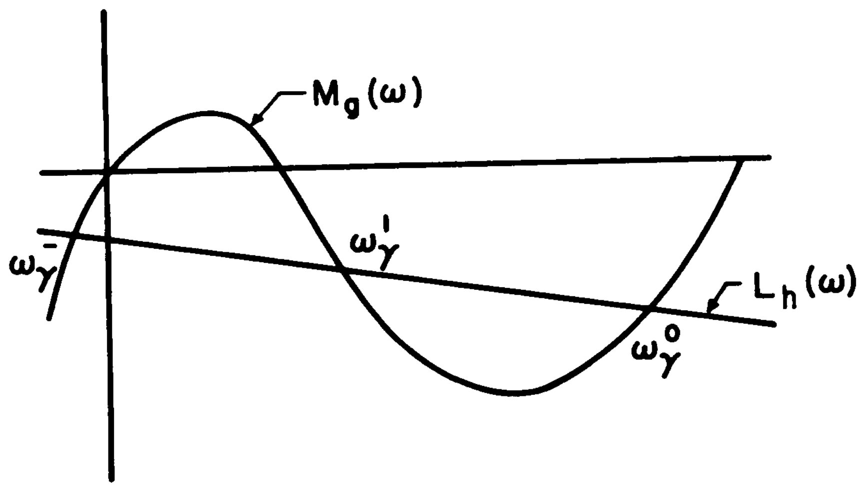

$M_g(\omega)$
ω_γ^-
ω_γ^1
ω_γ^0
$L_h(\omega)$

FIG. A3

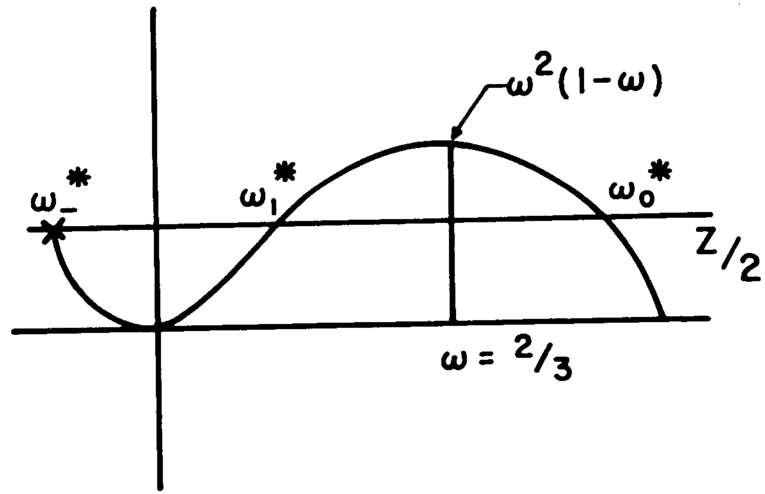

$\omega^2(1-\omega)$
ω_-^*
ω_1^*
ω_0^*
$Z/2$
$\omega = 2/3$

FIG. A4

while if the roots of G are complex, A45 is satisfied for all real ω. Also as before, ω_f^o and ω_f^1 must be real, and A46 is satisfied for

$$\omega_f^o \geq \omega \geq \omega_f^1 \tag{A49}$$

Again, combining A47, A48 and A49 for the case of real ω_g^o and ω_g^1, the physical solutions of A1 must satisfy

$$\omega_f^o \geq \omega \geq \omega_g^o \tag{A50}$$

$$\omega_g^1 \geq \omega \geq \omega_f^1 \tag{A51}$$

so that it must be shown that the positive roots of G lie inside the positive roots of F.

The proof here is different from that in the previous case, since the functions F and G have two parameters (A and Z) rather than only one, as F_1 and G_1 have. Here

$$G - F = 3\omega^2(1-\omega) - \frac{3}{2} Z \tag{A52}$$

which is greater than zero for

$$\omega \leq \omega_-' \qquad \omega_1' \leq \omega \leq \omega_o'$$

where ω_-', ω_1' and ω_o' are the three roots of

$$3\omega^2(1-\omega) - \frac{3}{2} Z = 0$$

(see Fig. A4).

Then, in order for the positive roots of G to lie between the positive roots of F, it must be true that

$$\omega_1' < \omega_f^1 \qquad (A53)$$

and

$$\omega_1' < \omega_g^1 \qquad (A54)$$

or, since F(0) and G(0) are both greater than zero

$$F(\omega_1') = G(\omega_1') > 0 \qquad (A55)$$

By substitution in equation A40

$$F_\gamma(\omega_1') = Z\left[\frac{1 - (g-2h)\omega_1'}{2\omega_1'}\right] + A\omega_1'$$

or

$$F_\gamma(\omega_1') = Z\left[\frac{1 - 3\omega_1'}{2\omega_1'}\right] + A\omega_1' \qquad (A56)$$

since $g - 2h = 3$ for all γ. $F_\gamma(\omega_1')$ is thus certainly greater than zero if

$$\omega_1' \leq \frac{1}{3} \qquad (A57)$$

Now, at $\omega = \frac{1}{3}$, $\omega^2(1-\omega) = \frac{2}{27}$ and from Fig. A4, A57 will be true if $Z \leq \frac{4}{27}$.

From equation 3.9:

$$\theta(1 + 2D\epsilon) = D\left[(1-\omega)^2 + A - \frac{Z}{\omega}\right] \qquad (A58)$$

it can be seen that if a physical pre-shock state exists

$$(1-\omega^{o})^{2} + A - \frac{Z}{\omega^{o}} > 0 \tag{A59}$$

where ω^{o} is the value of ω for the pre-shock state. From A59

$$Z \leq [\omega(1-\omega)^{2}]_{max} + \omega^{o}A \tag{A60}$$

where $[\omega(1-\omega)^{2}]_{max}$ is the maximum value of $\omega(1-\omega)^{2}$ between 0 and 1:

$$[\omega(1-\omega)^{2}]_{max} = \frac{4}{27}$$

Then

$$Z \leq \frac{4}{27} + \omega^{o}A \tag{A61}$$

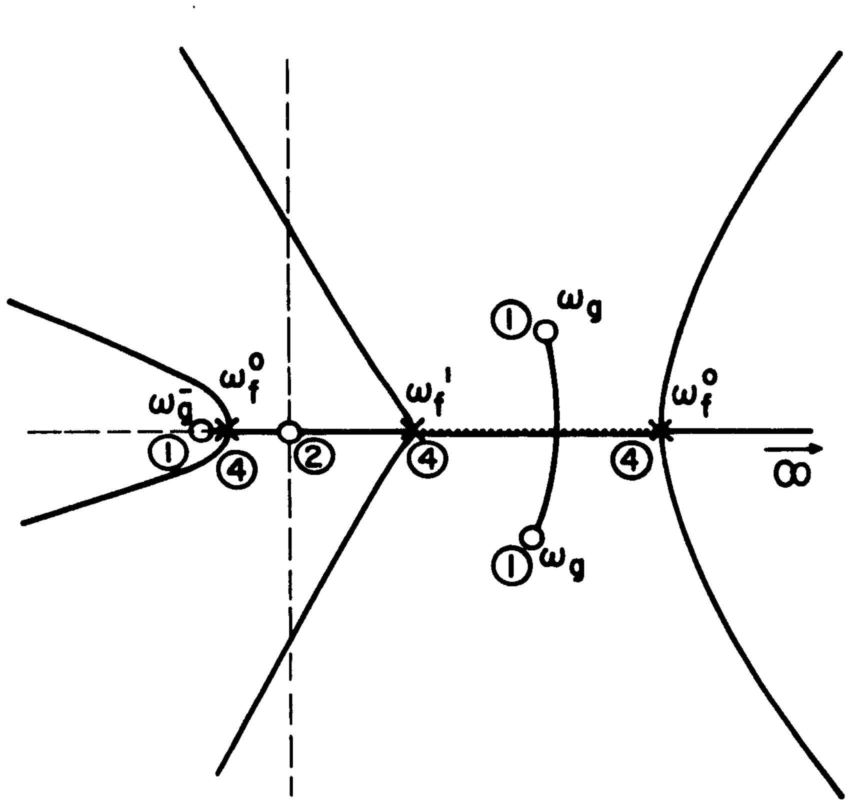
Z and A are two constants which depend on the four independent constants m, P, E and Q. Therefore Z and A are independent of each other, and for a particular Z A61 must hold for all possible A, in particular for $A = 0$. Then

$$Z \leq \frac{4}{27} \tag{A62}$$

and A57 holds, so that the positive roots of G lie between the positive roots of F.

Equation A1 can now be analyzed by the root locus method. Its left-hand side has 5 zeros, and 12 poles. It is possible to define regions R, or R_o and R_1 in the appropriate case, as before, and describe all possible root and pole combinations in two Figs. A5 and A6. (Note, these figures are drawn for $\omega_g^- < \omega_f^-$. For the purposes of this discussion the location of ω_g^- relative to ω_f^- is irrelevant.) The conclusions drawn from these diagrams are identical to the corresponding conclusions in the previous case, and the general result is that for every pre-shock state there exists one and only one post-shock state.

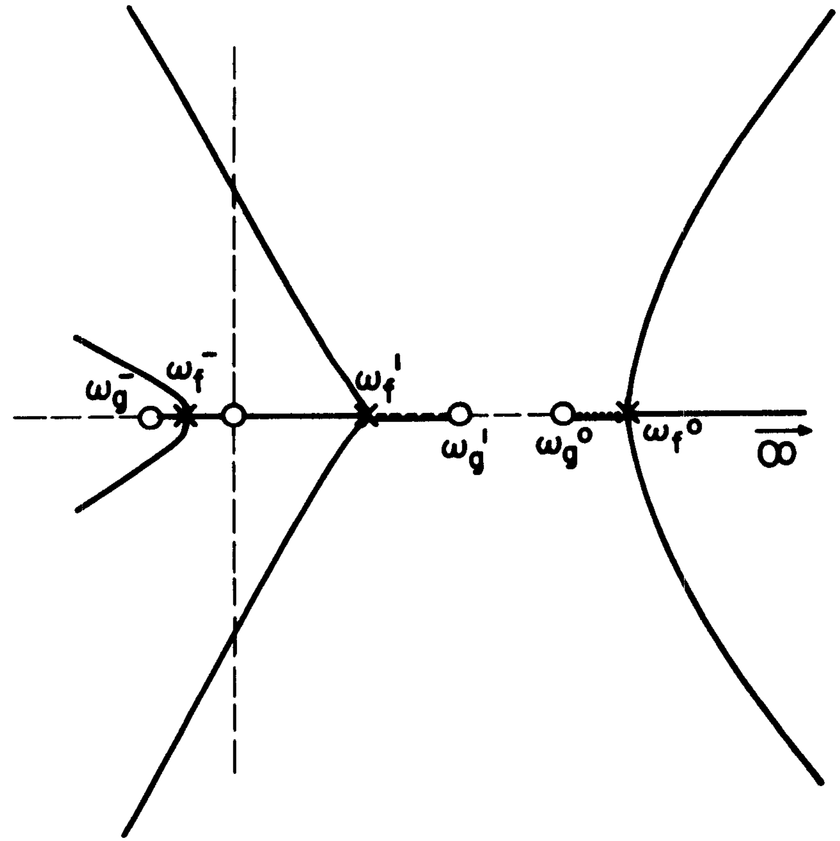
The results of this Appendix can be summarized graphically in Fig. 3.1, which is described in Chapter 3.



ⓝ ORDER OF POLE OR ZERO

········ REGION R —— ROOT LOCI

FIG. A5



········ REGION R_0 ------REGION R_1

FIG. A6

APPENDIX B

SIGNAL PROPAGATION SPEED

In this appendix the signal speed in a gas under the physical conditions described in Chapter 2 is derived using the standard methods of gas dynamics. To do this, the time-dependent mass flow, momentum flow and combined Maxwell equations (ignoring viscous effects and assuming infinite conductivity) are written:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho v = 0 \tag{B1}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(p + p_r + \frac{\mu H^2}{2} + \rho v^2) = 0 \tag{B2}$$

$$\frac{\partial H}{\partial t} = - \frac{\partial}{\partial x} vH \tag{B3}$$

combining equations B1 and B2

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial}{\partial x} (p + p_r + \frac{\mu H^2}{2}) \tag{B4}$$

Equations B1, B3, and B4 can be linearized by assuming small disturbances about equilibrium:

$$p_t = p_{to} + p' \tag{B5}$$

$$\rho = \rho_o + \rho'$$

$$v = v' \tag{B7}$$

$$H = H_o + h' \tag{B8}$$

where the total pressure:

$$p_t = p + p_r \tag{B9}$$

the sum of the thermodynamic and radiation pressures, and it is assumed that the fluid is stationary, i.e., that there is no velocity in the large. Assuming p', ρ', v' and h' to be first order infinitesimals, and p_{to}, ρ_o and H_o to be independent of x and t, the equations become, to first order:

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v'}{\partial x} = 0 \tag{B10}$$

$$\frac{\partial v'}{\partial t} = - \frac{1}{\rho_o} \frac{\partial}{\partial x}(p' + \mu H_o h') \tag{B11}$$

$$\frac{\partial h'}{\partial t} = - H_o \frac{\partial v'}{\partial x} \tag{B12}$$

Differentiating B11 and B12 will respect to t and x respectively and combining:

$$\frac{\partial^2 v'}{\partial t^2} = - \frac{1}{\rho_o} \frac{\partial^2 p'}{\partial x \partial t} + \frac{\mu H_o^2}{\rho_o} \frac{\partial^2 v'}{\partial x^2} \tag{B13}$$

If it is assumed that the disturbance is adiabatic, then

$$\frac{\Delta p'}{\Delta \rho'} = \frac{dp_t}{d\rho}\Big|_S \equiv \alpha^2 \tag{B14}$$

i.e., α^2 is the change in total pressure due to a change in density at constant entropy. The Alfvén velocity (Ref. 47)

$$\beta^2 \equiv \frac{\mu H_o^2}{\rho_o} \tag{B15}$$

so that B13 can be rewritten:

$$\frac{\partial^2 v'}{\partial t^2} = -\frac{\alpha^2}{\rho_o}\frac{\partial^2 \rho'}{\partial x \partial t} + \beta^2 \frac{\partial^2 v'}{\partial x^2} \tag{B16}$$

If B10 is differentiated with respect to x and combined with B16 the result is

$$\frac{\partial^2 v'}{\partial t^2} = (\alpha^2 + \beta^2)\frac{\partial^2 v'}{\partial x^2} \equiv v_o^2 \frac{\partial^2 v'}{\partial x^2} \tag{B17}$$

which is a wave equation with propagation velocity

$$v_o^2 = \alpha^2 + \beta^2 \tag{B18}$$

In order to determine α^2 in terms of the thermodynamic variables, the gas can be considered as obeying the equation of state:

$$p_t = \rho RT + \frac{1}{3} aT^4 \tag{B19}$$

with energy per unit mass

$$U = \frac{RT}{\gamma - 1} + \frac{aT^4}{\rho} \tag{B20}$$

From the combined first and second laws of thermodynamics:

$$TdS = dU - \frac{p_t}{\rho^2}\, d\rho \tag{B21}$$

when the entropy is constant,

$$\left.\frac{dU}{d\rho}\right|_S - \frac{p_t}{\rho^2} = 0 \tag{B22}$$

Using equation B20

$$\frac{dU}{d\rho} - \frac{p_t}{\rho^2} = 0 = \left(\frac{R}{\gamma - 1} + \frac{4aT^3}{\rho}\right)\frac{dT}{d\rho} - \frac{(p_t + aT^4)}{\rho^2} \tag{B23}$$

where all the derivatives are at constant entropy. Then

$$\frac{dT}{d\rho} = \frac{T}{\rho}\,\frac{(1 + \frac{4}{3}\,\epsilon)}{(\frac{1}{\gamma - 1} + 4\epsilon)} \tag{B24}$$

where

$$\epsilon \equiv \frac{aT^3}{\rho R} \tag{B25}$$

From B19

$$\frac{dp_t}{d\rho} = \rho R(1 + \frac{4}{3}\,\epsilon)\frac{dT}{d\rho} + RT = \alpha^2 \tag{B26}$$

so that

$$\alpha^2 = RT\left(1 + \frac{(1 + \frac{4}{3}\epsilon)^2}{(\frac{1}{\gamma - 1} + 4\epsilon)}\right) \tag{B27}$$

and the signal speed

$$v_o = \sqrt{\alpha^2 + \beta^2} = \sqrt{\gamma' RT + \mu H^2/\rho} \tag{B28}$$

where

$$\gamma' = \left(1 + \frac{(1 + \frac{4}{3}\epsilon)^2}{(\frac{1}{\gamma - 1} + 4\epsilon)}\right) \tag{B29}$$

is the equivalent specific heat ratio for signal speed.

APPENDIX C

ANALYSIS OF THE SHOCK STRUCTURE EQUATIONS

This appendix deals in general with equations of the form

$$L \frac{dy}{dx} = F(y) \tag{C1}$$

where L is an $n \times n$ diagonal matrix,

y is an n dimensional column vector,

$F(y)$ is an n dimensional line vector,

and equation C1 is to be integrated between two points y_o and y_1, such that

$$F(y_o) = F(y_1) = 0 \tag{C2}$$

The analysis is later specialized to the cases of two equations, namely, equations 5.19 and 5.20, and of four equations, namely, the equations of the first moment approximation.

Points y_o and y_1 are called equilibrium points of C1, and it is clear that any direct integration of C1 from either point y_o or y_1 will fail. For instance, if x_o is the value of x for which $y = y_o$, then by Taylor's theorem

$$y(x_o + \Delta x) = y_o + \frac{dy}{dx}\Big|_{x_o} \Delta x + \frac{d^2y}{dx^2}\Big|_{x_o} \frac{\Delta x^2}{2!} + \dots \tag{C3}$$

$$y(x_o + \Delta x) = y_o \tag{C4}$$

for all finite Δx since all other terms in the series will be proportional to $F(y_o)$. Then x_o and x_1, the value of x for which $y = y_1$, must both be infinite in absolute value, and a method must be found for the integration of C1 other than a direct one.

Equation C1 can be most conveniently treated in its "phase plane," which is an n-dimensional space with Cartesian coordinates $y^1 \dots y^n$, and its behavior in the neighborhoods of the points in this space y_o and y_1 must be studied (Refs. 16 and 17). In these neighborhoods

$$F(y_{0,1} + \Delta y) = M\Delta y + \quad (\Delta y^2) \tag{C5}$$

where M is the $n \times n$ matrix

$$M_{ij} = \frac{\partial F_i}{\partial y^j} \tag{C6}$$

evaluated at the particular point (y_o or y_1). In each of these neighborhoods in turn a change of origin can be performed, so that for each equilibrium point the linearization of C1 can be written as

$$L \frac{dy}{dx} = My \tag{C7}$$

The formal solution of equation C7 is

$$y = \sum_{k=1}^{n} A_k e^{\lambda_k x} \tag{C8}$$

where each A_k is a column vector, and the n λ_k's are the solutions of the equation

$$\det | M - \lambda L | = 0 \tag{C9}$$

To show that this is the case, each term in C8 can be seen to satisfy C7, i.e., by substitution

$$L\lambda_k A_k e^{\lambda_k x} = MA_k e^{\lambda_k x} \tag{C10}$$

or

$$(M - \lambda_k L)A_k = 0 \tag{C11}$$

which is true for arbitrary A_k if and only if C9 is satisfied. Thus, in the neighborhood of an equilibrium point, solutions of C1 have an exponential dependence on x given by C8.

It has been shown that x_o and x_1 must be infinite in absolute value, i.e., equal to either + or $-\infty$. There can be two distinct directions of integration of C1, either from y_o to y_1, in which case $x_o = -\infty$ and $x_1 = +\infty$ or from y_1 to y_o, in which case $x_1 = -\infty$ and $x_o = +\infty$. However, the nature of equation C1 is such that only one, if any, of the two directions of integration will give a unique solution curve of C1 which passes through both y_o and y_1.

At each equilibrium point in the phase plane, call the exiting direction that for which its corresponding x is $-\infty$ and the entering direction that for which its corresponding x is $+\infty$. It is clear that in C8 certain of the terms will apply to the exiting direction, namely, those whose λ_k are > 0, while the rest of the terms, namely, those for which $\lambda_k < 0$, will apply to the entering direction. Let

r_o = the number of positive λ_k's at y_o

s_o = the number of negative λ_k's at y_o

r_1 = the number of positive λ_k's at y_1

s_1 = the number of negative λ_k's at y_1.

Then, the exiting direction at y_o occupies an r_o dimensional manifold, i.e., if x_o corresponds to $-\infty$, the integral curve of C1 in the phase plane can leave y_o in any direction which lies within an r_o dimensional plane at y_o, determined by the eigenvectors corresponding to the r_o positive λ_k's. Similarly, the entering direction at y_o occupies an s_o dimensional manifold, and the exiting and entering directions at y_1 occupy r_1 and s_1 dimensional manifolds respectively. Call the direction of integration from y_o to y_1 the forward direction and from y_1 to y_o the backward direction. Then it is clear that in the forward direction, the solution curve of C1 occupies an r_o dimensional manifold at y_o and an s_1 dimensional manifold at y whereas, in the backward direction, the curve occupies an s_o dimensional manifold at y_o and an r_1 dimensional manifold at y_1. Then an integral curve in the forward direction lies along the intersection of an r_o dimensional manifold and an s_1 dimensional manifold, while an integral curve in the backward direction lies along the intersection of an r_1 dimensional manifold and an s_o dimensional manifold in n dimensional space.

The dimensionality of the intersection of a p dimensional manifold with a q dimensional manifold in an n dimensional space is given by

$$n - (n - p) - (n - q) = p + q - n$$

so that for a unique integral curve (a one-dimensional manifold) to exist in the forward direction it must be true that

$$r_o + s_1 = n + 1 \qquad \text{(C12)}$$

and for such a curve to exist in the backward direction it must hold that

$$r_1 + s_o = n + 1 \tag{C13}$$

It is clear that C12 and C13 cannot both be true at once, since

$$r_o + s_o = r_1 + s_1 = n \tag{C14}$$

so that there is at most one direction of integration for which a unique integral curve of C1 exists.

<u>The Eddington Approximation</u>

In this case $n = 2$ and the two equations corresponding to C1 are

$$L_P \frac{d\omega}{dx} = \omega + \frac{\theta}{\omega} + \frac{1}{3} S\theta^4 - 1 = F(\omega, \theta) \tag{C15}$$

$$L_Q \frac{d\theta}{dx} = \theta - D[(1-\omega)^2 + A] + 2DS\omega\theta^4 = G(\omega, \theta) \tag{C16}$$

When $n = 2$, a simplification and extension of the above analysis is possible (Ref. 17). In the first place, it is clear that a unique solution curve to C1 will exist if at one of the equilibrium points both λ's have the same sign (such a point is called a node), and at the other point the λ's are of opposite signs (such a point is called a saddle point). The integral curve in this case will be the intersection of a two-dimensional manifold (the entire phase plane) with a one-dimensional manifold, and the slope of the integral curve in the phase plane at the saddle point can be determined.

If at the saddle point λ^s is the eigenvalue of opposite sign to that of the λ's at the node, the solution to C7 near the saddle point corresponding to λ^s can be written:

$$y^1 = y^{1,s} e^{\lambda^s x} \tag{C17}$$

$$y^2 = y^{2,s} e^{\lambda^s x} \tag{C18}$$

and, substituting into C7

$$L_{11}\lambda^s y^1 = M_{11}y^1 + M_{12}y^2 \tag{C19}$$

$$L_{22}\lambda^s y^2 = M_{21}y^1 + M_{22}y^2 \tag{C20}$$

so that

$$(M_{11} - \lambda^s L_{11})y^1 + M_{12}y^2 = 0 \tag{C21}$$

$$M_{21}y^1 + (M_{22} - \lambda^s L_{22})y^2 = 0 \tag{C22}$$

Since the determinant of the coefficients of C21 and C22 is zero, they have a common solution, and the ratio of y^2 to y^1 is given by:

$$\frac{y^2}{y^1} = - \frac{M_{11} - \lambda^s L_{11}}{M_{12}} = \frac{-M_{21}}{M_{22} - \lambda^s L_{22}} \qquad \text{(C23)}$$

But equation C7 has been obtained by a change of origin such that the saddle point corresponds to $y^1 = y^2 = 0$. Thus $\frac{y^2}{y^1}$ is actually the slope of the solution curve at the saddle point, and

$$\frac{dy^2}{dy^1} = - \frac{M_{11} - \lambda^s L_{11}}{M_{12}} = \frac{-M_{21}}{M_{22} - \lambda^s L_{22}} \qquad \text{(C24)}$$

at the saddle point. Thus, in the case of $n = 2$, equation C1 can be integrated numerically in the phase plane from the saddle point to the node using the quotient equation:

$$\frac{L_{22}}{L_{11}} \frac{dy^2}{dy^1} = \frac{F_2(y)}{F_1(y)} \qquad \text{(C25)}$$

the slope of the solution curve to C25 at the saddle point being determined by C24.

The above method can be applied to equations C15 and C16. The components of the matrix M are

$$M_{11} = F_\omega = 1 - \frac{\theta}{\omega^2} \tag{C26}$$

$$M_{12} = F_\theta = \frac{1}{\omega} + \frac{4}{3} S\theta^3 = \frac{1}{\omega}(1 + \frac{4}{3}\epsilon) \tag{C27}$$

$$M_{21} = G_\omega = 2D(S\theta^4 + (1-\omega)) \tag{C28}$$

$$M_{22} = G_\theta = 1 + 8DS\omega\theta^3 = 1 + 8D\epsilon \tag{C29}$$

The equation for the eigenvalues λ can be written

$$L_P L_Q \lambda^2 - (L_Q F_\omega + L_P G_\theta)\lambda + (F_\omega G_\theta - F_\theta G_\omega) = 0 \tag{C30}$$

The sum of the eigenvalues at either y_o or y_1 is thus

$$\lambda_1^{o,1} + \lambda_2^{o,1} = \frac{F_\omega}{L_P} + \frac{G_\theta}{L_Q} \tag{C31}$$

and their product is

$$\lambda_1^{o,1}\lambda_2^{o,1} = \frac{1}{L_P L_Q}(F_\omega G_\theta - F_\theta G_\omega) \tag{C32}$$

The term in parenthesis in C32 is equal to

$$(1 + 8D\epsilon)(1 - \frac{1}{M^2})$$

since

$$G_\omega = 2D\,\frac{\theta}{\omega}\left(1 + \frac{4}{3}\,\epsilon\right) \tag{C33}$$

using the jump equations, and thus

$$F_\omega G_\theta - F_\theta G_\omega = (1 + 8D\epsilon)\left[1 - \frac{\theta}{\omega^2}\left(1 + \frac{2D\left(1 + \frac{4}{3}\,\epsilon\right)^2}{1 + 8D\epsilon}\right)\right] \tag{C34}$$

$$F_\omega G_\theta - F_\theta G_\omega = (1 + 8D\epsilon)\left[1 - \frac{\gamma'\theta}{\omega^2}\right] \tag{C35}$$

Then

$$\lambda_1^{o,1}\lambda_2^{o,1} = \frac{1}{L_P L_Q}(1 + 8D\epsilon_{0,1})\left[1 - \frac{1}{M_{0,1}^2}\right] \tag{C36}$$

Now $M_o > 1$ and $M_1 < 1$ so that

$$\lambda_1^o\lambda_2^o > 0 \tag{C37}$$

and

$$\lambda_1^1\lambda_2^1 < 0 \tag{C38}$$

Point y_o is thus a node, and y_1 is a saddle point. To determine whether the positive or negative λ^1 is to be used, it can be seen from C31 that

$$\lambda_1^o + \lambda_2^o > 0 \tag{C39}$$

since F_ω and G_θ are both positive at that point. Then λ_1^o and λ_2^o are both positive, and the negative eignevalue at y_1 is to be used. The integral curve is thus exiting at y_o and entering at y_1, x_1 corresponding to $+\infty$. The numerical integrations, however, must be carried on from $x = +\infty$ to $x = -\infty$, since the slope at y_1 is known, given by

$$\left.\frac{d\theta}{d\omega}\right|_{y_1} = - \frac{F_\omega - \lambda_1^1 L_P}{F_\theta} = - \frac{G_\omega}{G_\theta - \lambda_1^1 L_Q} \tag{C40}$$

λ_1^1 being the negative eigenvalue at y_1.

The First Moment Approximation

In this case $n = 4$, the four equations corresponding to C1 being

$$L_P \frac{d\omega}{dx} = \omega + \frac{\theta}{\omega} + \frac{1}{3}\phi_o - 1 \equiv F_1(y) \tag{C41}$$

$$L_H \frac{d\theta}{dx} = \theta - 2D[(1-\omega)^2 + A] + 2D\omega\phi_o + \frac{2D\phi_1}{3\beta} \equiv F_2(y) \tag{C42}$$

$$L_R \frac{d\phi_o}{dx} = - \phi_1 \equiv F_3(y) \tag{C43}$$

$$\frac{1}{3} L_R \frac{d\phi_1}{dx} = S\theta^4 - \phi_o \equiv F_4(y) \tag{C44}$$

It will be shown here that a unique solution curve to C41 - C44 exists. The matrix M is

$$M = \begin{pmatrix} 1 - \frac{\theta}{\omega^2} & \frac{1}{\omega} & \frac{1}{3} & 0 \\ \frac{2D\theta}{\omega}(1 + \frac{4}{3}\epsilon) & 1 & 2D\omega & \frac{2D}{3\beta} \\ 0 & 0 & 0 & -1 \\ 0 & \frac{4\epsilon}{\omega} & -1 & 0 \end{pmatrix} \tag{C45}$$

so that C9 becomes

$$\begin{vmatrix} 1 - \frac{\theta}{\omega^2} - \lambda L_P & \frac{1}{\omega} & \frac{1}{3} & 0 \\ \frac{2D\theta}{\omega}(1 + \frac{4}{3}\epsilon) & 1 - \lambda L_H & 2D\omega & \frac{2D}{3\beta} \\ 0 & 0 & -\lambda L_R & -1 \\ 0 & \frac{4\epsilon}{\omega} & -1 & \frac{-\lambda L_R}{3} \end{vmatrix} = 0 \tag{C46}$$

The determinant in C46 is more conveniently expressed as the sum of two determinants, equation C46 becoming

$$\begin{vmatrix} (\gamma'-1-\gamma' L_P) & \frac{1}{\omega} & \frac{1}{3} & 0 \\ 2D\omega(1 + \frac{4}{3}\epsilon) & 1-\lambda L_H & 2D\omega & \frac{2D}{3\beta} \\ 0 & 0 & -\lambda L_R & -1 \\ 0 & \frac{4\epsilon}{\omega} & -1 & \frac{-\lambda L_R}{3} \end{vmatrix}$$

$$+ (1 - \frac{1}{M^2}) \begin{vmatrix} 1 & \frac{1}{\omega} & \frac{1}{3} & 0 \\ -2D\omega(1 + \frac{4}{3}\epsilon) & 1-\lambda L_H & 2D\omega & \frac{2D}{3\beta} \\ 0 & 0 & -\lambda L_R & -1 \\ 0 & \frac{4\epsilon}{\omega} & -1 & \frac{-\lambda L_R}{3} \end{vmatrix} = 0 \tag{C47}$$

where $\frac{1}{M^2} = 1 - \frac{\gamma'\theta}{\omega^2}$.

The determinants in C47 can be expanded to give the following equation in λ:

$$\lambda^4 - \frac{1}{L_P}\left(\frac{\gamma'-1}{\gamma'} + \frac{L_P}{L_H}\right)\lambda^3 - \frac{1}{L_P L_H}\left(\frac{\gamma'-1}{\gamma'} \frac{4(\gamma - \frac{4}{3})}{1 + \frac{4}{3}\epsilon} + \frac{3L_Q L_P}{L_R^2}\right)\lambda^2 +$$

$$+ \frac{3L_Q}{L_R^2 L_P L_H}\left(\frac{\gamma'-1}{\gamma'} + \frac{L_P}{L_Q}(1 + 8D\epsilon)\right)\lambda - \left(1 - \frac{1}{M^2}\right)\left[\frac{\lambda^3}{\gamma' L_P} - \right.$$

$$\left. - \frac{1}{L_P L_H \gamma'}\left(\gamma + \frac{8}{3} D\epsilon\right)\lambda^2 - \frac{3L_Q}{\gamma' L_P L_R^2 L_H}\lambda + \frac{3(1 + 8D\epsilon)}{L_P L_R^2 L_H}\right] = 0 \qquad \text{(C48)}$$

where

$$L_Q = L_H + \frac{8D\epsilon}{3\beta\omega} L_R \qquad \text{(C49)}$$

Equation C48 can be written

$$\gamma' L_P\left(1 - \frac{1}{M^2}\right) = \frac{\lambda[\lambda^3 - a_2\lambda^2 - a_1\lambda + a_o]}{\lambda^3 - b_2\lambda^2 - b_1\lambda + b_o} \equiv \frac{\lambda p(\lambda)}{q(\lambda)} \qquad \text{(C50)}$$

where all the coefficients a_i and b_i are positive and

$$p(\lambda) = \lambda^3 - a_2\lambda^2 - a_1\lambda + a_o \qquad \text{(C51)}$$

$$q(\lambda) = \lambda^3 - b_2\lambda^2 - b_1\lambda + b_o \qquad \text{(C52)}$$

By Descarte's rule of signs p and q have one real negative root each. The root locus method (see Appendix A) can now be applied to equation C50, with $\gamma' L_p(1 - \frac{1}{M^2})$, the root locus parameter, being positive for state 0 and negative for state 1. The root loci for states 0 and 1 are shown in Figs C1 and C2 respectively. From Figs. C1 and C2, in which the singularities are not distinguished as to roots and poles, it is clear that there are two negative roots of C48 at P_1, i.e., $s_1 = 2$, and that there are three roots with positive real part at P_o, i.e., $r_o = 3$. Thus

$$r_o + s_1 = 5 = n + 1 \tag{C53}$$

and a shock curve exists.

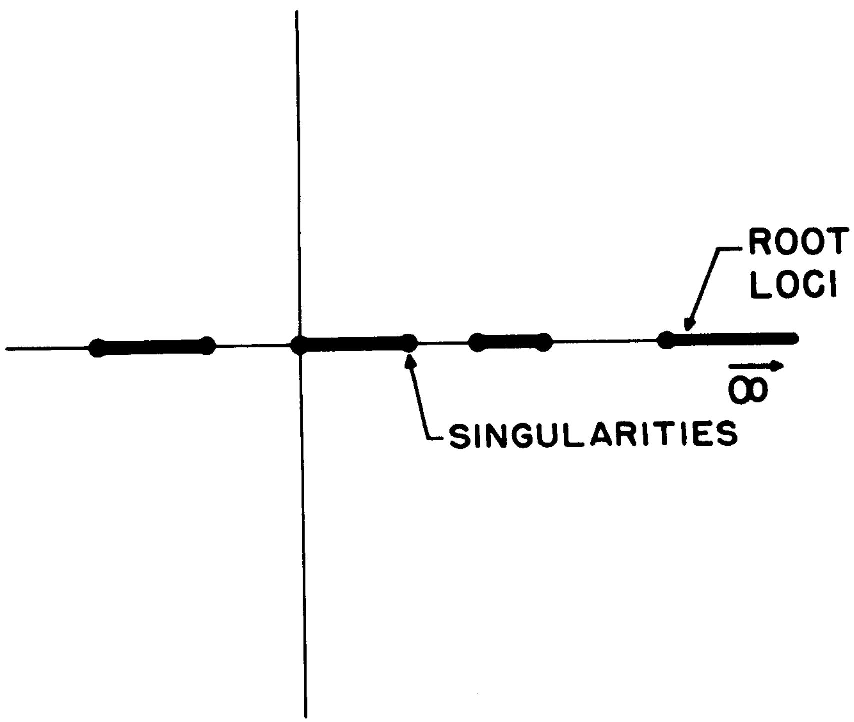


FIG. C1

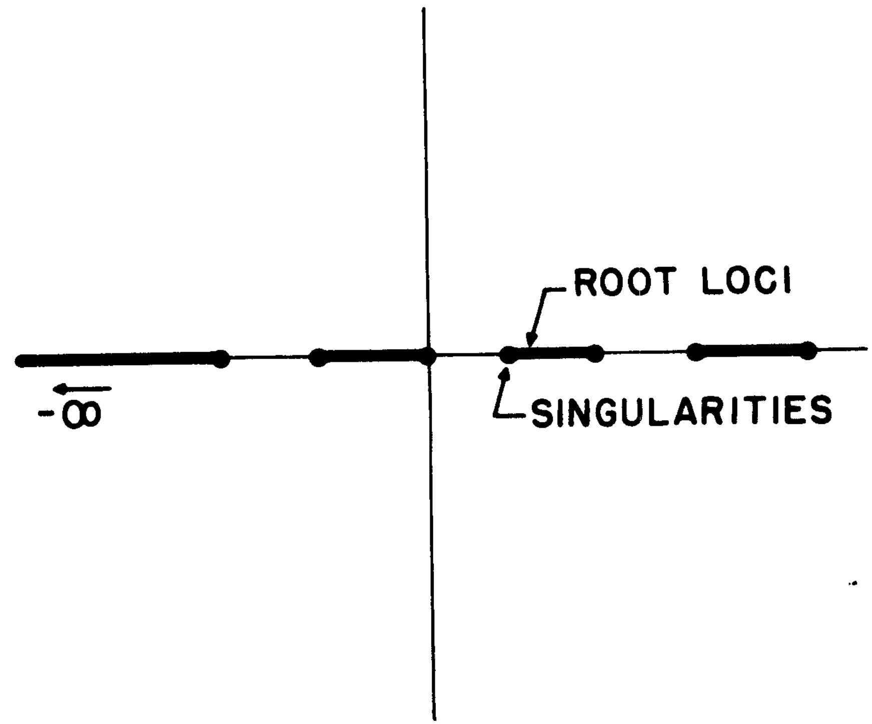


FIG. C2

APPENDIX D

SOME NUMERICAL RESULTS

The purpose of this appendix is the presentation of the numerical procedures and results pertinent to the analyses and conclusions in the rest of this paper. Numerical studies were made of 45 cases, as listed in Table D1, with the following ranges of parameters:

Initial Mach Number $50 \geq M_o \geq 2$

Initial Radiation Parameter $10 \geq \epsilon_o \geq 0.1$

Initial Temp. in ^oK $10^6 \geq T_o \geq 10^4$.

In all of the above, the magnetic field was zero. No cases with non-zero magnetic field were considered numerically. In Table D1, the cases are divided into 15 groups of three, denoted by the Roman numbers I - XV, the cases included in each of these groups having the same values of the dimensionless parameters M_o and ϵ_o. Many of the results presented below depend only on these parameters, so that only 15 cases need be considered. Table D1 includes, besides the initial values of M, ϵ, and T for each case, also the final values of these three variables.

There were four numerical problems which had to be solved in connection with this paper. These were:

(1) Solution of the jump equation,

(2) Determination of the shock curve,

(3) Finding the maximum value of $G(\omega, \theta)$ along the shock curve, and

(4) Finding the region of validity of the Eddington approximation.

Each of these problems will be treated separately below.

Solution of the Jump Equation

This was essentially a simple algebraic problem, and an IBM 1620 Fortran program was written for its solution. This program is listed in Table D2. The inputs to this program are the initial conditions M_o, ϵ_o and T_o, and the outputs are the final conditions M_1, ϵ_1 and T_1 as well as the values of velocity, density, thermodynamic pressure and total pressure at the initial and final states. The actual solution of the jump equation is accomplished in a subroutine labeled "Ranhug," in which the location of the post-shock state on the ω axis is approximated by the method of Fig. 3.1, and Newton's method is then used to find the value of ω^1 as closely as desired. The results for the post-shock state are partially tabulated in Table D1, and are shown graphically in Figs. 3.2 - 3.7.

The Shock Curve

Thanks to the analysis in Chapter V, the determination of the shock curve is reduced from a problem in numerical integration to an algebraic one with negligible loss of accuracy. An IBM 7090 Fortran program was devised which, using as inputs the outputs of the jump-equation program,

(1) Determines the values of the characteristic lengths L_P, L_Q and L_R at the initial and final states.

(2) Finds the values of $\overline{L}_P$, $\overline{L}_Q$, $\overline{L}_R$ and δ

(3) Finds the sign of F_ω at point P_1 and thus classifies the problem according to whether or not the Eddington approximation is valid,

(4) Determines 20 - 30 points along the shock curve in the phase plane using the methods of the Eddington approximation whether or not it is valid, and

(5) Finds the shock thicknesses t_ω and t_θ, using the results of (4), and compares them to $\overline{L}_R$. This program is listed in Table D3.

It is assumed in Chapter V that the parameter $\delta = \dfrac{\overline{L}_P}{\overline{L}_Q}$ is very small. That this is the case is demonstrated in Table D4, which lists the values of $\overline{L}_P$, $\overline{L}_R$, $\overline{L}_Q$ and δ for each of the 45 cases. Also listed in Table D4 are $\dfrac{\overline{L}_R}{t_\omega}$ and $\dfrac{\overline{L}_R}{t_\theta}$ for each case. (It is to be noted that the meaning of the latter results is doubtful in cases where the Eddington approximation does not apply.) Table D5 is a classification of the 15 groups according to the validity of the Eddington approximation. In Figs. 5.6 and 5.7 phase plane plots of the results of (4) are shown for typical cases in which the Eddington approximation does and does not apply, respectively, and the corresponding curves of T and v vs x are shown in Fig. 5.9. Figure D1 shows how the temperature shock thickness varies with M_o for various values of ϵ_o. The values of $t_{\theta\ ref}$ for use with this figure are given in Table D6.

The Maximum Value of $G(\omega, \theta)$

A necessary step in the argument that the Eddington approximation holds when inequalities 5.48 and 5.49 are satisfied is the statement that the maximum possible value of $G(\omega, \theta)$ along the shock curve is of order 1. The validity of that statement was tested for cases I - XV using a simple numerical procedure, the evaluation of $G(\omega, \theta)$ along the curve $F(\omega, \theta) = 0$. It can be shown that the maximum of $G(\omega, \theta)$ in the allowed region:

$$F(\omega, \theta) \leq 0 \tag{D1}$$

$$G(\omega, \theta) \geq 0 \tag{D2}$$

is indeed along the curve $F(\omega, \theta) = 0$. If an arbitrary path in the phase plane

$$\Delta\omega = h \tag{D3}$$

$$\Delta\theta = k \tag{D4}$$

is chosen, the change in G along that path is

$$\Delta G = G_{\omega} h + G_{\theta} k \tag{D5}$$

Since

$$G_{\omega} = 2D(S\theta^{4} + (1-\omega)) \tag{D6}$$

$$G_{\theta} = 1 + 8D\omega S\theta^{3} \tag{D7}$$

and both are greater than zero, ΔG will be positive along a path such that

$$h > 0 \tag{D8}$$

$$k > 0 \tag{D9}$$

Then, from Figs. 5.3 and 5.4 it is clear that the maximum value G_{max} of $G(\omega, \theta)$ in the region defined by D1 and D2 is along $F(\omega, \theta) = 0$, and therefore the maximum value of G along the shock curve must be less than or equal to G_{max}. the Values of G_{max} for cases I - XV are listed in Table D7, from which it is clear that for those cases $G(\omega, \theta)$ along the shock curve does not exceed one.

Region of Validity of the Eddington Approximation

The criterion for the validity of the Eddington approximation is

$$1 - \frac{\theta^{1}}{\omega^{1}}{}^{2} > 0 \tag{D10}$$

and therefore the curve

$$\frac{\theta^1}{{\omega^1}^2} = 1 \qquad \text{(D11)}$$

is the boundary between two regions in the $\theta^1 - \omega^1$ plane, in one of which the approximation is valid and in the other one of which it is not. The curve D11 can be transferred to the $\theta^0 - \omega^0$ plane by means of the jump equation. This is accomplished by means of an IBM 1620 Fortran program which is listed in Table D8. In this program, curve D11 is found for

$$\frac{4 - \sqrt{6}}{10} \leq \omega^1 \leq \frac{1}{2} \qquad \text{(D12)}$$

$$\frac{11 - 4\sqrt{6}}{50} \leq \theta^1 \leq \frac{1}{4} \qquad \text{(D13)}$$

and the corresponding values of θ^0 and ω^0 are calculated by means of the subroutine RHIN wherein Newton's method is used on the jump equation A18.

The origin of the limits in D12 is as follows: Besides satisfying D11, θ^1 and ω^1 must satisfy the jump equations, and must also both be greater than zero. The strength parameters A and S must also be greater than zero. In particular, the following must be satisfied:

$$F(\omega^1, \theta^1) = \omega^1 + \frac{\theta^1}{\omega^1} + \frac{1}{3} S{\theta^1}^4 - 1 = 0 \qquad \text{(D14)}$$

$$(3\theta^1) = -(1-\omega^1)(1-7\omega^1) - A \qquad \text{(D15)}$$

Equations D14 and D15 can both be written in terms of ω^1 as follows:

$$\frac{1}{3} S{\omega^1}^8 + 2\omega^1 - 1 = 0 \qquad \text{(D16)}$$

$$10{\omega^1}^2 - 8\omega^1 + 1 = -A \qquad \text{(D17)}$$

Since S and A must both be greater than or equal to zero, D16 can only be satisfied if

$$2\omega^1 - 1 \leq 0$$

or

$$\omega^1 \leq \frac{1}{2} \qquad \text{(D18)}$$

and D17 can only be satisfied if

$$10{\omega^1}^2 - 8\omega^1 + 1 \leq 0$$

or

$$\frac{4-\sqrt{6}}{10} \leq \omega^1 \leq \frac{4+\sqrt{6}}{10} \qquad \text{(D19)}$$

Inequality D12 is derived from D18 and D19.

The results of the calculation described in this section are shown in Fig. 5.8.

TABLE D1

INITIAL AND FINAL STATES

Case No.	M_o	ϵ_o	$T_o(^oK)$	M_1	ϵ_1	$T_1(^oK)$	Rom. Des.
1	2	0.1	10^4	0.598	0.278	1.88×10^4	
2	2	0.1	10^5	0.598	0.278	1.88×10^5	I
3	2	0.1	10^6	0.598	0.278	1.88×10^6	
4	2	1	10^4	0.580	1.54	1.59×10^4	
5	2	1	10^5	0.580	1.54	1.59×10^5	II
6	2	1	10^6	0.580	1.54	1.59×106	
7	2	10	10^4	0.570	11.50	1.47×10^4	
8	2	10	10^5	0.570	11.50	1.47×10^5	III
9	2	10	10^6	0.570	11.50	1.47×10^6	
10	5	0.1	10^4	0.429	1.92	4.46×10^4	
11	5	0.1	10^5	0.429	1.92	4.46×10^5	IV
12	5	0.1	10^6	0.429	1.92	4.46×10^6	
13	5	1	10^4	0.409	5.12	2.98×10^4	
14	5	1	10^5	0.409	5.12	2.98×10^5	V
15	5	1	10^6	0.409	5.12	2.98×10^6	
16	5	10	10^4	0.398	25.42	2.41×10^4	
17	5	10	10^5	0.398	25.42	2.41×10^5	VI
18	5	10	10^6	0.398	25.42	2.41×10^6	
19	10	0.1	10^4	0.382	6.71	7.34×10^4	
20	10	0.1	10^5	0.382	6.71	7.34×10^5	VII
21	10	0.1	10^6	0.382	6.71	7.34×10^6	
22	10	1	10^4	0.372	14.82	4.53×10^4	
23	10	1	10^5	0.372	14.82	4.53×10^5	VIII
24	10	1	10^6	0.372	14.82	4.53×10^6	
25	10	10	10^4	0.366	64.27	3.47×10^4	
26	10	10	10^5	0.366	64.27	3.47×10^5	IX
27	10	10	10^6	0.366	64.27	3.47×10^6	
28	20	0.1	10^4	0.364	20.53	1.10×10^5	
29	20	0.1	10^5	0.364	20.53	1.10×10^6	X
30	20	0.1	10^6	0.364	20.53	1.10×10^7	
31	20	1	10^4	0.360	42.75	6.61×10^4	
31	20	1	10^5	0.360	42.75	6.61×10^5	XI
31	20	1	10^6	0.360	42.75	6.61×10^6	
32	20	10	10^4	0.357	176.96	4.95×10^4	
33	20	10	10^5	0.357	176.96	4.95×10^5	XII
34	20	10	10^6	0.357	176.96	4.95×10^6	
37	50	0.1	10^4	0.356	83.93	1.79×10^5	
38	50	0.1	10^5	0.356	83.93	1.79×10^6	XIII
39	50	0.1	10^6	0.356	83.93	1.79×10^7	
40	50	1	10^4	0.355	170.97	1.06×10^5	
41	50	1	10^5	0.355	170.97	1.06×10^6	XIV
42	50	1	10^6	0.355	170.97	1.06×10^7	
43	50	10	10^4	0.354	696.5	7.86×10^4	
44	50	10	10^5	0.354	696.5	7.86×10^5	XV
45	50	10	10^6	0.354	696.5	7.86×10^6	

TABLE D2

```
*1605
      COMMON COV,COT,CT3,SQCT,AS
  100 FORMAT (F12.5,2(4X,E12.5),5X,I1)
  101 FORMAT (E12.5,4X,E12.5,I5)
  102 FORMAT (8HCASE NO.,I3)
  103 FORMAT (I3)
  107 FORMAT (3H PT,4X,1HM,16X,5HTHETA,12X,5HOMEGA,12X,7HEPSILON)
  108 FORMAT (I5,4(2X,E15.8))
  109 FORMAT (3H PT,4X,8HVELOCITY,7X,11HTEMPERATURE,4X,7HDENSITY,8X,13HT
     1HERM. PRESS.,2X,11HTOT. PRESS.   )
  110 FORMAT (I5,4(2X,E13.6),1X,E13.6)
  112 FORMAT (2(2X,E15.8),15X,30X,1H1)
  113 FORMAT (4E15.8,19X,1H1)
      READ  103, IDIN
      READ 101, CC,DEL,KC1
      GAMMA = 5./3.
      R = 16.634E7
      DE = 1./3.
      DE2 = 2.*DE
      SB = 7.67E-15
      ID = IDIN
    6  READ 100, EM1,EPS1,T1,JJ
      PUNCH 102, ID
      HO1 = SB*T1**3/(EPS1*R)
      GP1 = 1.+(1.+4./3.*EPS1)**2./(1./DE2+4.*EPS1)
      U1 = SQRTF(GP1*R*T1)*EM1
      P1 = HO1*R*T1
      PT1 = P1*(1.+EPS1/3.)
      AS = HO1*U1
      QS = AS*U1+PT1
      ES = QS*U1+AS*(R*T1*(1./DE2+EPS1)-U1*U1/2.)
      AL = 2.*ES*AS/(QS*QS)-1.
      BET = QS/(2.0E10*AS)
      COV = AS/QS
      COT = R*COV*COV
      W1 = COV*U1
      H1 = COT*T1
      S = EPS1/(W1*H1**3)
      IF (SENSE SWITCH 1) 18,27
   18 PAUSE
   27 GO TO (19,26),JJ
   19 CALL RANHUG(DE,S,AL,W1,CC,W2,H2,J)
   26 IF (SENSE SWITCH 1) 20,21
   20 PAUSE
   21 GO TO (10,6),J
   10 U2 = W2/COV
      T2 = H2/COT
      EPS2 = S*W2*H2**3
      GP2 = 1.+(1.+4./3.*EPS2)**2./(1./DE2+4.*EPS2)
      EM2 = U2/SQRTF(GP2*R*T2)
      HO2 = AS/U2
      P2 = HO2*R*T2
      PT2 = P2*(1.+EPS2/3.)
      CT3 = COT*COT*COT
      SQCT = SQRTF(COT)
      C2P = CLR(H2,W2)
      CP0 = CLR(H1,W1)
      PUNCH 107
      K = 0
      PUNCH 108, K,EM1,H1,W1,EPS1
```

```
      K = 1
      PUNCH 108, K,EM2,H2,W2,EPS2
      PUNCH 109
      K = 0
      PUNCH 110, K,U1,T1,HO1,P1,PT1
      K = 1
      PUNCH 110, K,U2,T2,HO2,P2,PT2
      PUNCH 113, EM1,EM2,AL,S
      PUNCH 113, W1,W2,H1,H2
      PUNCH 113, COV,COT,BET,AS
      PUNCH 112, C2P,CRO
      ID = ID+1
      GO TO 6
      END
*1605
      SUBROUTINE RANHUG(DA,SA,AA,A1,CA,A2,A3,N)
      DIMENSION C(10),F(10)
      N = 1
      D = 6.-1./DA
      B = 3.*D**3/SA
      C(1) = 2401.
      C(2) = -10976.
      C(3) = 20188.+1372.*AA
      C(4) = -(19040.+4704.*AA)
      C(5) = 9766.+5964.*AA+294.*AA*AA
      C(6) = -(2720.+3392.*AA+672.*AA*AA)
      C(7) = 412.+852.*AA+468.*AA*AA+28.*AA*AA*AA
      C(8) = -(32.+96.*AA*(1.+AA)+32.*AA*AA*AA+B*(7.-D))
      C(9) = (1.+AA)**4+(8.-D)*B
      C(10) = -(1.+AA)*B
       U = 7.-D
      IF (D) 14,13,12
   12 WB = (4.-SQRTF(9.-7.*AA))/7.
      DIS = (U-1.)**2-4.*U*AA
      IF (DIS) 15,16,16
   15 WF = (4.+SQRTF(9.-7.*AA))/7.
      GO TO 17
   16 WF = (U+1.-SQRTF(DIS))/(2.*U)
      GO TO 17
   14 WB = (U+1.-SQRTF((U-1.)**2-4.*U*AA))/(2.*U)
      DIS = 9.-7.*AA
      IF (DIS) 18,18,19
   18 WF = (U+1.+SQRTF((U-1.)**2-4.*U*AA))/(2.*U)
      GO TO 17
   19 WF = (4.-SQRTF(DIS))/7.
   17 WI = (WF-WB)/10.
    5 W2T = WB
      F(1) = C(1)
      DO 20 I = 2,10
   20 F(I) = C(I)+F(I-1)*W2T
      R = F(10)
   23 W2T = W2T+WI
      IF (WE-W2T) 3,3,4
    3 WI = WI/10.
      GO TO 5
    4 F(1) = C(1)
      DO 8 I = 2,10
    8 F(I) = C(I) +F(I-1)*W2T
      IF (R*F(10)) 1,10,22
   22 R = F(10)
```

TABLE D2 (Cont'd.)

```
      GO TO 23
   13 RETURN
    1 FP = 0.0
      DO 9 I = 1,9
      XI = 10-I
    9 FP = FP+XI*C(I)*W2T**(9-I)
      DX = F(10)/FP
      W2T = W2T-DX
      IF (ABSF(DX)-CA) 10,10,11
   10 A2 = W2T
      A3 = ((1.-A2)*(7.*A2-1.)-AA)/D
      RETURN
   11 F(1) = C(1)
      DO 2 I = 2,10
    2 F(I) = C(I)+F(I-1)*W2T
      GO TO 1
      END
*1605
      FUNCTION CLR(H,W)
      COMMON COV,COT,CT3,SQCT,AS
      T = H/COT
      T1 = 2.035E-3*T**(1./3.)
      IF (0.5-T1) 1,1,2
    1 CON = 1.+T1
      GO TO 3
    2 CON = 1./(1.-T1)
    3 CLR  = .6009E-3 *H*H*H*SQRTF(H)*W*W/(AS*AS*CT3*SQCT*COV*COV)*CON
      RETURN
      END
```

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TABLE D3

```
      FOOF(TH1,OM1)=TH1/OM1+OM1+(S*TH1*TH1*TH1*TH1)/3.-1.
      GOOF(TH2,OM2)=TH2-D*((1.-OM2)*(1.-OM2)+A)+2.*D*OM2*S*TH2*TH2*TH2*
     1TH2
      FOMF(TH1  , OM1  ) = 1.-TH1  /(OM1  *OM1  )
      TOFF(TH2  ) = TH2  /CT
      VOFF(OM2  ) = OM2  /CV
      EPSF(TH3  ,OM3  ) = S*OM3  *TH3  *TH3  *TH3
      GAMPF(F) = 1.+2.*(1.+4.*F/3.)**2/(3.+8.*F)
      PSIF (V,T) = .9802E-15*V*T*T*T/FM
      GTHF(H,W) = 1.+8.*D*EPSF(H,W)
      GOMF (H,W) = 2.*D*(1.-W+S*H*H*H*H)
      FTHF(H,W) = (1.+4.*EPSF(H,W)/3.)/W
  104 FORMAT (1H0,22H OUT OF ALLOWED REGION)
  105 FORMAT (1H0, 18HINDETERMINATE FORM)
  106 FORMAT (1H0, 12H F = 0 CURVE)
  986 FORMAT (1H0,11HINCR. ('/2),I3)
  998 FORMAT (F6.1,2I5)
  997 FORMAT (1H0,5HTV/LP,12X,7HTV/M*LP)
  108 FORMAT (1H0,6HTHETS=E15.8,4X,6HOMEGS=E15.8)
   95 FORMAT (5I5)
  100 FORMAT (1H0,2HPT 7X,1HX 16X,5HOMEGA 12X,5HTHETA 12X,1HF 16X,1HG
     116X,1HM)
  111 FORMAT (1H ,1H*,I4,6(2X,E15.8))
  101 FORMAT (1H ,I5,6(2X,E15.8))
  102 FORMAT (1H0,6HMIN. F,11X,7HVEL.TH.,10X,7HRAT-VEL,10X,9HAT/ OMEGA,8
     1X,5HTHETA)
  103 FORMAT (1H ,E15.8,4(2X,E15.8))
 1102 FORMAT (1H0,6HMAX. G,11X,8HTEMP.TH.,9X,8HRAT-TEMP,9X,9HAT/ OMEGA,8
     1X,5HTHETA)
 1103 FORMAT (1H0,14HMAX. DDPHI/PHI,3X,15HMAX.2TERM/1TERM,3X,9HAT/ OMEGA
     1,7X,5HTHETA)
 1104 FORMAT (1H ,E15.8,3(2X,E15.8))
   86 FORMAT (1H1 8HCASE NO. I3)
  999 FORMAT (1H )
   85 FORMAT (4F15.8)
  996 FORMAT (1H ,E15.8,2X,E15.8)
   96 FORMAT (1H0 7X,3HM0= E15.8,7X,3HM1= E15.8,8X,2HA= E15.8,8X 2HS=
     1E15.8)
   97 FORMAT (1H 4X,6HOMEG0= E15.8, 4X,6HOMEG1= E15.8,4X,6HTHET0= E15.8,
     14X,6HTHET1= E15.8)
  297 FORMAT (1H 7X,3HCV= E15.8,7X,3HCT= E15.8,5X,5HBETA= E15.8,8X,2HM=
     1E15.8)
  194 FORMAT (4(2X,E15.8))
  195 FORMAT (2(2X,E15.8))
  196 FORMAT (1H0 4X,5HLP(1) 12X,5HLP(0) 12X,5HLR(1) 12X,5HLR(0) 12X,
     15HLQ(1) 12X,5HLQ(0))
  197 FORMAT (1H  6(2X,E15.8))
  190 FORMAT(3F15.8)
  191 FORMAT (1H0 3X,7HLAMBDA= E15.8,4X,6HTHETS= E15.8,4X,6HOMEGS=
     1E15.8)
  192 FORMAT (2F15.8)
  193 FORMAT (1H 4X,6HLP(S)= E15.8,4X,6HLR(S)= E15.8)
   98 FORMAT (1H0 2HPT 4X, 1HM 16X 1HT 16X 3HEPS 14X 1HV)
   99 FORMAT (1H  I2, 4(2X, E15.8))
  910 FORMAT (11H ONLY CURVE)
 1999 FORMAT (1H0,14HOUTSIDE CURVES)
 3999 FORMAT (1H ,I5,1H*,I5)
 1000 FORMAT  ( 2H , 95X,6HFW = 0)
 1001 FORMAT (1H ,95X,7HFW POS.)
 1002 FORMAT (1H ,95X,7HFW NEG.)
```

TABLE D3 (Cont'd.)

```
  112 FORMAT ( 1H0,2X,2HPT 5X,1HX 14X,8HVELOCITY 7X,11HTEMPERATURE 5X,7H
     1DENSITY 8X,12HTHERM.PRESS. 3X,10HTOT.PRESS. 6X,1HM 14X7HEPSILON)
  113 FORMAT  (1H ,I5,8(1X,E14.7))
 9999 FORMAT (1H0,7HMEAN LP,10X,7H MEAN LP,10X,7HMEAN LQ,10X,5HDELTA)
 9998 FORMAT (1H ,F15.8,3(2X,F15.8))
      READ INPUT TAPE 5, 95, IDIN,IDOUT,KIN,KE,K1E
      READ INPUT TAPE 5, 998, ZC,JOE,KHE
      ID = IDIN
      D = 1./3.
   18 WRITE OUTPUT TAPE 6, 86, ID
      READ INPUT TAPE 5, 999
      READ INPUT TAPE 5, 85, UM0, UM1, A, S
      WRITE OUTPUT TAPE 6, 96, UM0, UM1, A, S
      READ INPUT TAPE 5, 85, OMEG0, OMEG1, THET0, THET1
      WRITE OUTPUT TAPE 6, 97, OMEG0, OMEG1, THET0, THET1
      READ INPUT TAPE 5, 85, CV, CT, BET, EM
      WRITE OUTPUT TAPE 6,297, CV, CT, BET, EM
      READ INPUT TAPE 5,195,CRY1,CRYZ
      PS = PSIF (OMEG0/CV,THET0/CT)
      IF (PS-1.0E-4) 2001,2002,2002
 2001 PSL = PS-PS*PS/2.
      GO TO 2003
 2002 PSL = LOGF(1.+PS)
 2003 CON = (THET0/CT)**2.5/PSL
      ETA = .382E-14*CON
      C1Z = 4.*ETA/(3.*EM)
      CH0 = 80.*C1Z
      PS = PSIF (OMEG1/CV,THET1/CT)
      IF (PS-1.0E-4) 2004,2005,2005
 2004 PSL = PS-PS*PS/2.
      GO TO 2006
 2005 PSL = LOGF(1.+PS)
 2006 CON = (THET1/CT)**2.5/PSL
      ETA = .382E-14*CON
      C1  = 4.*ETA/(3.*EM)
      CH1 = 80.*C1
      COF = 8.*S/(9.*BET)
      C2Z = CH0+COF*CRYZ*THET0*THET0*THET0
      C2  = CH1+COF*CRY1*THET1*THET1*THET1
      WRITE OUTPUT TAPE 6,196
      WRITE OUTPUT TAPE 6,197,C1,C1Z,CRY1,CRYZ,C2,C2Z
      SQCT = SQRTF(CT)
      CT3 = CT*CT*CT
      CLPM = SQRTF(C1)*SQRTF(C1Z)
      CLRM = SQRTF(CRY1)*SQRTF(CRYZ)
      CLQM = SQRTF(C2)*SQRTF(C2Z)
      DE = CLPM/CLQM
      WRITE OUTPUT TAPE 6, 9999
      WRITE OUTPUT TAPE 6, 9998, CLPM,CLRM,CLQM,DE
      R = CT/(CV*CV)
      T = TOFF(THET0)
      V = VOFF (OMEG0)
      EP = EPSF(THET0, OMEG0)
      WRITE OUTPUT TAPE 6, 98
      K = 0
      WRITE OUTPUT TAPE 6, 99, K,UM0,T,EP, V
      T = TOFF(THET1)
      V = VOFF(OMEG1)
      EP = EPSF(THET1, OMEG1)
      K = 1
```

TABLE D3 (Cont'd.)

```
      WRITE OUTPUT TAPE 6, 99, K,UM1, T, EP,V
      WRITE OUTPUT TAPE 6, 112
      FW1 = FOMF(THET1,OMEG1)
      IF (FW1) 1,2,3
    1 H = THET1
      WRITE OUTPUT TAPE 6, 1002
      WM = SQRTF(H)
      WNC = 0.1*(WM-OMEG1)
      FAC = 1.-S*H*H*H*H/3.
      W2 = 0.5*(FAC+SQRTF(FAC*FAC-4.*H))
      G0 = GOOF(H,W2)
      H0 = -FOMF(H,W2)/FTHF(H,W2)
      VC = DF*G0/H0
      J = 1
      X = 0.
      KC = 1
      FN = 0.
      GN = 0.
      W = OMEG1+WNC
      FL = 0.
   13 F = FOOF(H,W)
      GO TO (4,5),J
    5 IF (F-VC) 4,6,6
    4 IF (W-WM) 7,7,8
    8 J = 2
    7 G = GOOF(H,W)
      IF (FN-F) 9,9,10
   10 FN = F
      HFN = H
      WFN = W
    9 IF (GN-G) 11,12,12
   11 GN = G
      HGN = H
      WGN = W
   12 FB = 0.5*(F+FL)
      X = X-CLPM*WNC/FB
      FL = F
      EM = W/SQRTF(GAMPF(EPSF(H,W))*H)
      T = TOFF(H)
      V = VOFF(W)
      RH = EM/V
      P = RH*R*T
      E = EPSF(H,W)
      PT = P*(1.+E/3.)
      WRITE OUTPUT TAPE 6, 113, KC,X,V,T,RH,P,PT,EM,E
      W = W+WNC
      KC = KC+1
      GO TO 13
    6 HNC = 0.1*(THET1-THET0)
      W = W2
      F = VC
      G = GOOF(H,W)
      IF (GN-G) 14,15,15
   14 GN = G
      HGN = H
      WGN = W
   15 FB = 0.5*(F+FL)
      GL = G
      X = X-CLPM*WNC/FB
      EM = W/SQRTF(GAMPF(EPSF(H,W))*H)
```

```
      T = TOFF(H)
      V = VOFF(W)
      RH = FM/V
      P = RH*R*T
      E = EPSF(H,W)
      PT = P*(1.+E/3.)
      WRITE OUTPUT TAPE 6, 113, KC,X,V,T,RH,P,PT,FM,E
      L = 1
      KC = KC+1
      GO TO 16
    3 HNC = 0.05*(THET1-THET0)
      WRITE OUTPUT TAPE 6, 1001
      H = THET1
      W = OMEG1
      GL = 0.
      L = 2
      X = 0.
      FN = 0.
      GN = 0.
      KC = 1
   16 H = H-HNC
      FAC = 1.-S*H*H*H*H/3.
      W = 0.5*(FAC+SQRTF(FAC*FAC-4.*H))
      G = GOOF (H,W)
      IF (GN-G) 17,17,19
   17 GN = G
      HGN = H
      WGN = W
   19 HO = -FOME(H,W)/FTHE(H,W)
      F = DF*G/HO
      GO TO (20,21),L
   21 IF (FN-F) 20,20,22
   22 FN = F
      HFN = H
      WFN = W
   20 GR = 0.5*(G+GL)
      X = X+CLOM*HNC/GR
      GL = G
      FM = W/SQRTF(GAMPF(EPSF(H,W))*H)
      T = TOFF(H)
      V = VOFF(W)
      RH = FM/V
      P = RH*R*T
      E = EPSF(H,W)
      PT = P*(1.+E/3.)
      WRITE OUTPUT TAPE 6, 113, KC,X,V,T,RH,P,PT,EM,E
      KC = KC+1
      IF (H-THET0) 23,23,16
    2 WRITE OUTPUT TAPE 6, 1000
      GO TO 3
   23 TH = (THET1-THET0)*CLOM/GN
      TW = (OMEG1-OMEG0)*CLPM/FN
      RAH = CLPM/TH
      RAW = CLPM/TW
      WRITE OUTPUT TAPE 6, 102
      WRITE OUTPUT TAPE 6, 103, FN,TW,RAW,WFN,HFN
      WRITE OUTPUT TAPE 6, 1102
      WRITE OUTPUT TAPE 6, 103, GN,TH,RAH,WGN,HGN
      FMB = SQRTF(UM0*UM1)
      TLP = TW/CLPM
```

TABLE D3 (Cont'd.)

```
      TMLP = TLP/EMB
      WRITE OUTPUT TAPE 6 , 997
      WRITE OUTPUT TAPE 6, 996, TLP,TMLP
      IF (IDOUT-ID) 24,24,25
   25 ID = ID+1
      GO TO 18
   24 CALL EXIT
      END
```

TABLE D4

Case No.	$\bar{L}_P$ cm	$\bar{L}_R$ cm	$\bar{L}_Q$ cm	δ	$\bar{L}_R/t_\omega$	$\bar{L}_R/t_\theta$
1	4.84×10^{-3}	3.74×10^{7}	7.71×10^{10}	6.27×10^{-14}	4.16×10^{8}	5.08×10^{-4}
2	4.84×10^{-4}	1.26×10^{5}	8.21×10^{7}	5.90×10^{-12}	1.40×10^{7}	1.61×10^{-3}
3	4.84×10^{-5}	4.61×10^{2}	9.51×10^{4}	5.09×10^{-10}	5.12×10^{5}	5.08×10^{-3}
4	3.16×10^{-2}	2.57×10^{9}	3.76×10^{13}	8.42×10^{-16}	1.26×10^{9}	1.80×10^{-4}
5	3.16×10^{-3}	8.61×10^{6}	3.99×10^{10}	7.93×10^{-14}	4.24×10^{7}	5.70×10^{-4}
6	3.16×10^{-4}	3.14×10^{4}	4.59×10^{7}	6.89×10^{-12}	1.54×10^{6}	1.80×10^{-3}
7	1.47×10^{-1}	2.11×10^{11}	1.59×10^{16}	9.26×10^{-18}	1.92×10^{-4}	1.57×10^{-4}
8	1.47×10^{-2}	7.08×10^{8}	1.69×10^{13}	8.74×10^{-16}	6.08×10^{-4}	4.96×10^{-4}
9	1.47×10^{-3}	2.57×10^{6}	1.94×10^{10}	7.60×10^{-14}	1.92×10^{-3}	1.57×10^{-3}
10	5.37×10^{-3}	8.89×10^{7}	2.68×10^{11}	2.01×10^{-14}	1.19×10^{9}	1.28×10^{-3}
11	5.37×10^{-4}	3.03×10^{5}	2.89×10^{8}	1.86×10^{-12}	4.04×10^{7}	4.05×10^{-3}
12	5.37×10^{-5}	1.15×10^{3}	3.47×10^{5}	1.55×10^{-10}	1.53×10^{6}	1.28×10^{-2}
13	2.68×10^{-2}	3.92×10^{9}	5.89×10^{13}	4.56×10^{-16}	2.32×10^{9}	5.98×10^{-4}
14	2.68×10^{-3}	1.33×10^{7}	6.30×10^{10}	4.26×10^{-14}	7.85×10^{7}	1.89×10^{-3}
15	2.68×10^{-4}	4.94×10^{4}	7.43×10^{7}	3.61×10^{-12}	2.93×10^{6}	5.98×10^{-3}
16	1.07×10^{-1}	2.52×10^{11}	1.60×10^{16}	6.72×10^{-18}	1.06×10^{-3}	6.44×10^{-4}
17	1.07×10^{-2}	8.50×10^{8}	1.71×10^{13}	6.30×10^{-16}	3.35×10^{-3}	2.04×10^{-3}
18	1.07×10^{-3}	3.14×10^{6}	1.99×10^{10}	5.39×10^{-14}	1.06×10^{-2}	6.44×10^{-3}
19	4.83×10^{-3}	1.68×10^{8}	5.33×10^{11}	9.07×10^{-15}	3.92×10^{8}	3.18×10^{-3}
20	4.83×10^{-4}	5.77×10^{5}	5.80×10^{8}	8.34×10^{-13}	1.35×10^{7}	1.01×10^{-2}
21	4.83×10^{-5}	2.26×10^{3}	7.18×10^{5}	6.73×10^{-11}	5.28×10^{5}	3.18×10^{-2}
22	2.21×10^{-2}	6.73×10^{9}	9.51×10^{13}	2.32×10^{-16}	3.44×10^{-3}	1.61×10^{-3}
23	2.21×10^{-3}	2.29×10^{7}	1.02×10^{11}	2.15×10^{-14}	1.09×10^{-2}	5.09×10^{-3}
24	2.21×10^{-4}	8.72×10^{4}	1.23×10^{8}	1.79×10^{-12}	3.44×10^{-2}	1.61×10^{-2}
25	8.29×10^{-2}	4.05×10^{11}	2.22×10^{16}	3.73×10^{-18}	3.48×10^{-3}	1.82×10^{-3}
26	8.29×10^{-3}	1.37×10^{8}	2.39×10^{13}	3.48×10^{-16}	1.10×10^{-2}	5.77×10^{-3}
27	8.29×10^{-14}	5.16×10^{6}	2.83×10^{10}	2.93×10^{-14}	3.48×10^{-2}	1.82×10^{-2}
28	3.91×10^{-3}	3.10×10^{8}	9.09×10^{11}	4.30×10^{-15}	2.29×10^{-2}	9.42×10^{-3}
29	3.91×10^{-4}	1.08×10^{6}	9.99×10^{8}	3.92×10^{-13}	7.24×10^{-2}	2.98×10^{-2}
30	3.91×10^{-5}	4.36×10^{3}	1.28×10^{6}	3.06×10^{-11}	2.29×10^{-1}	9.42×10^{-2}
31	1.72×10^{-2}	1.22×10^{10}	1.52×10^{14}	1.14×10^{-16}	1.08×10^{-2}	4.76×10^{-3}
32	1.72×10^{-3}	4.18×10^{7}	1.64×10^{11}	1.05×10^{-14}	3.42×10^{-2}	1.51×10^{-2}
33	1.72×10^{-4}	1.63×10^{5}	2.02×10^{8}	8.52×10^{-13}	1.08×10^{-1}	4.76×10^{-2}
34	6.31×10^{-2}	7.20×10^{11}	3.36×10^{16}	1.88210^{-18}	1.15×10^{-2}	5.48×10^{-3}
35	6.31×10^{-3}	2.46×10^{9}	3.63×10^{13}	1.74×10^{-16}	3.62×10^{-2}	1.73×10^{-2}
36	6.31×10^{-4}	9.39×10^{6}	4.38×10^{10}	1.44×10^{-14}	1.15×10^{-1}	5.48×10^{-2}
37	2.77×10^{-3}	6.98×10^{8}	1.70×10^{12}	1.64×10^{-15}	9.80×10^{-2}	4.31×10^{-2}
38	2.77×10^{-4}	2.46×10^{6}	1.89×10^{9}	1.47×10^{-13}	3.31×10^{-1}	1.36×10^{-1}
39	2.77×10^{-5}	8.91×10^{3}	2.16×10^{6}	1.28×10^{-11}	9.80×10^{-1}	4.31×10^{-1}
40	1.20×10^{-2}	2.72×10^{10}	2.75×10^{14}	4.38×10^{-17}	5.20×10^{-2}	2.17×10^{-2}
41	1.20×10^{-3}	9.45×10^{7}	3.02×10^{11}	3.98×10^{-15}	1.65×10^{-1}	6.86×10^{-2}
42	1.20×10^{-4}	3.81×10^{5}	3.85×10^{8}	3.13×10^{-13}	5.20×10^{-1}	2.17×10^{-1}
43	4.36×10^{-2}	1.60×10^{12}	5.98×10^{16}	7.29×10^{-19}	5.61×10^{-2}	2.47×10^{-2}
44	4.36×10^{-3}	5.51×10^{9}	6.51×10^{13}	6.69×10^{-17}	1.77×10^{-1}	7.81×10^{-2}
45	4.36×10^{-4}	2.17×10^{7}	8.11×10^{10}	5.38×10^{-15}	5.61×10^{-1}	2.47×10^{-1}

TABLE D5

CASE NO.	EDDINGTON APPROXIMATION	
	INVALID	VALID
I	X	
II	X	
III		X
IV	X	
V	X	
VI		X
VII	X	
VIII		X
IX		X
X		X
XI		X
XII		X
XIII		X
XIV		X
XV		X

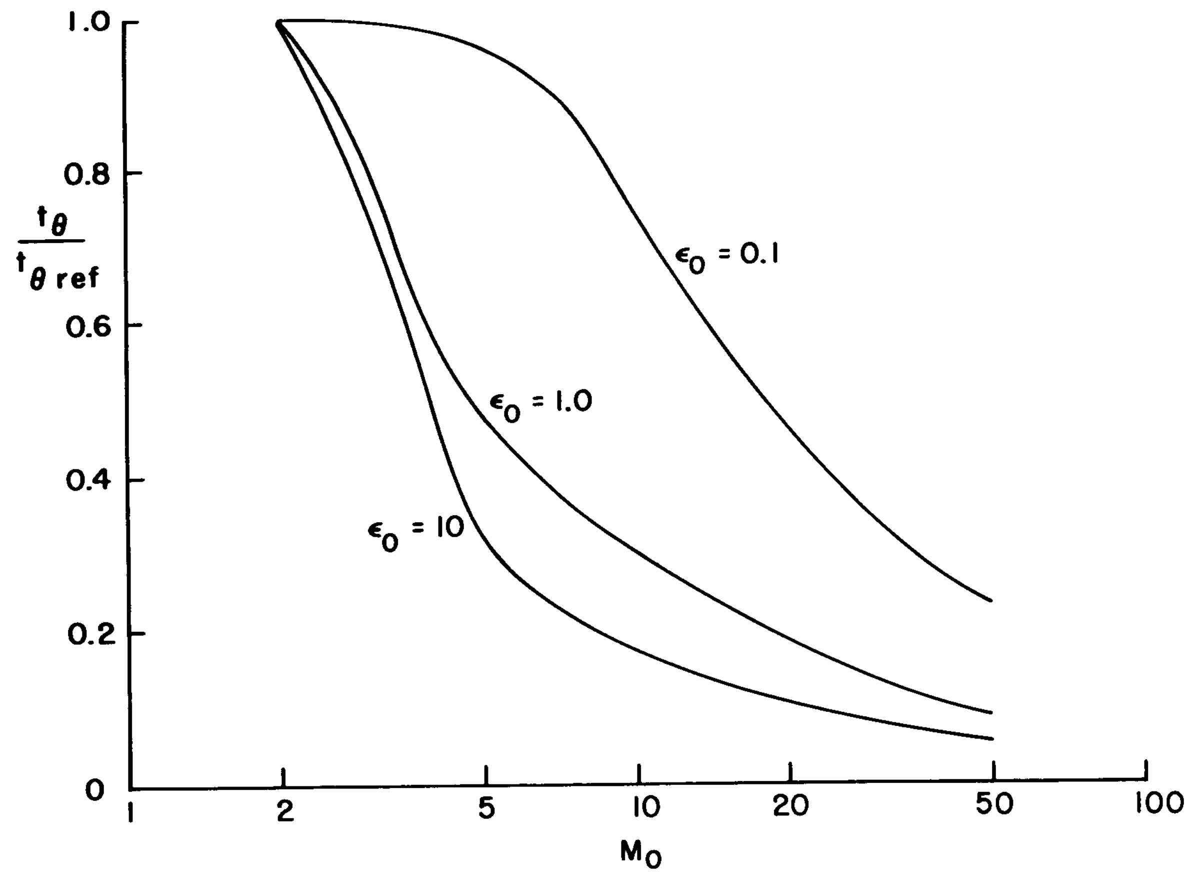

FIG. D1 TEMPERATURE THICKNESS vs. M_0 (VALUES OF $t_{\theta ref}$ IN TABLE D6)

TABLE D6

ϵ_o	T_o (^oK)	$t_{\theta\ ref}$ (cm)
0.01	10^4	7.4×10^{10}
	10^5	7.8×10^7
	10^6	9.1×10^4
1.0	10^4	1.4×10^{13}
	10^5	1.5×10^{10}
	10^6	1.7×10^7
10.0	10^4	1.3×10^{15}
	10^5	1.4×10^{12}
	10^6	1.6×10^9

TABLE D7

CASE NO.	$G_{max.}$
I	0.102
II	0.143
III	0.165
IV	0.304
V	0.339
VI	0.357
VII	0.378
VIII	0.397
IX	0.410
X	0.400
XI	0.416
XII	0.424
XIII	0.415
XIV	0.421
XV	0.425

TABLE D8

```
*1605
  100 FORMAT (I5,4X,E6.1)
  101 FORMAT (3HNO.,I5,5X,2HS=,E15.8,5X,2HA=,E15.8)
  104 FORMAT (3HPT.,3X,5HOMEGA,10X,5HTHETA,10X,4HEPS.,11X,7HG PRIME,8X,1
     1HM)
  102 FORMAT (I5,5(F15.7))
      READ 100, N,CC
      WB = 0.156
      F = 3.
      WE = 0.500
      EN = N
      WNC = (WE-WB)/(EN-1.)
      DO 1 I=1,N
      X = I-1
      W1 = WB+X*WNC
      H1 = W1*W1
      E1 = 3.*(1.-2.*W1)/W1
      S = E1/W1**7
      A = 8.*W1-10.*W1*W1-1.
      GPP1 = 1.+4.*E1/3.
      GP1 = 1.+GPP1*GPP1/(1.5+4.*E1)
      EM1 = 1./SQRTF(GP1)
      PUNCH 101, I,S,A
      PUNCH 104
      L = 1
      PUNCH 102, L,W1,H1,E1,GP1,EM1
      CALL RHIN(S,A,W1,W0,CC)
      H0 = ((1.-W0)*(7.*W0-1.)-A)/F
      E0 = S*W0*H0*H0*H0
      GPP2 = 1.+4.*E0/3.
      GP0 = 1.+GPP2*GPP2/(1.5+4.*E0)
      EM0 = SQRTF(W0*W0/(GP0*H0))
      L = 0
      PUNCH 102, L,W0,H0,E0,GP0,EM0
    1 CONTINUE
      END
*1605
      SUBROUTINE RHIN(SS,AA,D1,D0,RE)
      DIMENSION C(10),E(10)
      S = SS
      A = AA
      B = 81./S
      C(1) = 2401.
      C (2) = -10976.
      C(3) = 20188.+1372.*A
      C(4) = -(19040.+4704.*A)
      C(5) = 9766.+5964.*A+294.*A*A
      C(6) = -(2720.+3392.*A+672.*A*A)
      C(7) = 412.+852.*A+468.*A*A+28.*A*A*A
      C(8) = -(32.+96.*A*(1.+A)+32.*A*A*A+4.*B)
      C(9) = (1.+A)**4+5.*B
      C(10) = -(1.+A)*B
      IF (A-1.2857) 14,13,13
   13 WT = D1
      GO TO 7
   14 WB = (4.+SQRTF(9.-7.*A))/7.
      IF (A-.5625) 15,16,17
   15 WE = (5.+SQRTF(9.-16.*A))/8.
      GO TO 18
   16 WE = 0.625
```

TABLE D8 (Cont'd.)

```
      GO TO 18
   17 WF = D1+10.*RE
   18 WNC = (WE-WB)/10.
    4 WT = WB
      F(1) = C(1)
      DO 1 I = 2,10
    1 F(I) = C(I)+F(I-1)*WT
      R = F(10)
      IF(R) 6,7,6
    6 WT = WT+WNC
      IF (WF-WT) 2,3,3
    3 WNC = WNC/10.
      GO TO 4
    2 F(1) = C(1)
      DO 5 I = 2,10
    5 F(I) = C(I)+F(I-1)*WT
      IF (R*F(10)) 8,7,9
    9 R = F(10)
      GO TO 6
    7 DO = WT
      RETURN
    8 FP = 0.0
      DO 10 I = 1,9
      X = 10-I
   10 FP = FP+X*C(I)*WT**(9-I)
      DX = F(10)/FP
      WT = WT-DX
      IF (ABSF(DX)-RE) 7,7,11
   11 F(1) = C(1)
      DO 12 I = 2,10
   12 F(I) = C(I)+F(I-1)*WT
      GO TO 8
      END
```

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